# Consistency, Learning and Higher Order Beliefs: an empirical test

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**Abstract**

This paper studies the role of learning mechanisms, higher order beliefs and the sources of information agents use to form their expectations. According to a Bayesian process, agents' heterogeneous choices are aggregated, and then plugged into a standard macroeconomic framework. We investigate consistency between agents' maximization and the macroeconomic equilibrium confirming that higher order beliefs is a relevant factor for agents, while public information plays the most important role in determining individual expectations. Moreover, we show that the importance of higher order beliefs is not absolute but turns upon the type of information which agents face. Agents welcome any information that helps them to solve uncertainty about the state of the world, and this may reduce the role of higher order beliefs in the formation process. Estimation of the structural parameters is implemented by exploiting Foreign Exchange Consensus Survey data of heterogeneous forecasts and fundamentals from 2006 to 2012.

*Keywords:* higher order beliefs, exchange rates, economic fundamentals, survey data

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## 1. Introduction

A growing body of the recent empirical literature relates to the role information plays in the expectation formation mechanism using individual and aggregate data.[1] It is not yet clear, however, to what extent different information sources are relevant for the equilibrium relationships between economic aggregates and individual choices, and if heterogeneity among agents is important. At least some questions may still remain unanswered: Do people consider higher order beliefs, i.e., beliefs about other agents' strategies, when making their optimal choice? If so, how do people in higher order beliefs evaluate fundamentals?

This paper offers a rationale for the decision process that each individual takes in a market where individual expectations and fundamental variables defines the market equilibrium. We develop a simple dynamic framework *à* la Morris and Shin (2002) that highlights interactions between higher order beliefs and learning by taking into account the impact that individual's choices have on other participants. Learning helps agents in the decision about their own individual beliefs, but also to evaluate others' beliefs in an optimal manner.

Without loss of generality, we focus on the exchange rate market as an ideal environment where agents form their expectations by combining attention on economic fundamentals together with subjective knowledge on private information. The joint combination of public and private information is crucial in higher order beliefs when fundamentals play a role. One of the main difficulties in this framework is the observability of private information, hardly revealed by the simple observation of prices. Survey data on individual forecasts appear to be a reliable tool to recover private information for an econometric model. Our analysis is thus based on individual expectations obtained from the *Consensus Economic of London*, mainly looking at the *eur/usd* currency from 2006 to 2012.[2].

The first result this paper conveys to the literature of higher order beliefs is to provide an empirical framework that describes the expectation formation process. To the best of our

[1] These contributions are based on estimates of the process that drives expectations (see, e.g., Andolfatto et al., 2008 and Del Negro and Eusepi, 2011), as well as investigations of whether survey responses conform to various theories, e.g., sticky information theories (Reis, 2006; Branch, 2007).

[2] Some robustness with different exchange rates, i.e., *pound/usd* and *usd/yen* are reported in Section 6.

knowledge, this paper represents the first empirical test for a learning model of higher order beliefs. We demonstrate how public information, the transparency among investors, as well as macroeconomic variables can represent the main features of the expectations formation process. Our model examines how investors' choices should be optimally designed on the basis of the available information. Investors observe a public signal and exhibit common interest in coordination through higher order beliefs. Each agent also captures a private signal which can be interpreted as the investors' ability to correctly predict the real value of the fundamental and it is simply based on personal and subjective evaluation including opinions, rumors, economic projections and market commentary. The optimal design in this context thus depends on *i*) private information *ii*) other agents' beliefs *iii*) macroeconomic equilibrium. The structural model we propose is casted in a state-space form and estimated through Bayesian techniques. We measure the relevant information used by agents and disentangle the effects of both public and private signals identifying the weight assigned to each source of information.

The second contribution is to propose a characterization of internal consistency combining the learning process and the aggregate equilibrium. Investors in the exchange rate market may fail to recognize all explanatory variables needed to correctly predict exchange rates. This means that, independently by the received signals, each agent plays his optimal solution up to an error. We find conditions that aim at reducing this possible mistake. Taking into account a consistency condition in our model, indeed, implies agents remain rational all the way through. They act as *sophisticated* investors looking at an additional source of information, that in our setup is the market equilibrium, to estimate economic signals in a more precise way. They still choose their actions based on two components, i.e., fundamental motive and higher order beliefs as suggested above, but perceive that the uncertain state of the economy is relatively more complex and try to take it into account. We compare the model under consistency with the unrestricted model without consistency. In the latter case, agents, being more *naive*, choose their actions neglecting the role played by market equilibrium.

The empirical analysis provides some interesting results. First, when agents do not benefit from observing market equilibrium, i.e., when consistency is not taken into account, the weight assigned to higher order beliefs is extremely high. This means that each agent maximizes his/her utility largely based on others' choices, and this counts on forecasting decisions for about 90%. Our empirical result is consistent with Morris and Shin (2002) who study a static coordination game in the spirit of Keynes (1936) demonstrating how agents' choices are mainly influenced by a public source of information. This result is induced by the role of higher order beliefs which selectively reduce the importance of highly informative private information.[3] Our estimates show that public information plays the most important role in determining individual expectations with more than 94% of the aggregate information. Information publicly revealed does exert a disproportionate influence with respect to the private one. People reason in higher orders and consequently react to higher order beliefs they form. This becomes especially apparent in how investors respond to private and public signals, and it is true even if the precision of the private signal is larger than the public one, implicitly suggesting that the role of private information is crucial for a correct prediction of the model.

Second, in the model with consistency, agents benefit from observing market equilibrium. In this case, we show that the impact of higher order beliefs is reduced. The weight assigned to higher order beliefs is around 33%, while the weight of public information decreases to about 53%. The reason is that consistency relies on agents who evaluate their choice by taking into account the evaluation of the market equilibrium. Investors in this case have higher perception of the real economic environment and this is confirmed by higher values of signal precisions of the consistent model compared to the unrestricted one. Beliefs about fundamentals should guide investors' strategies. This means that each of these pieces of information is captured by the macroeconomic equilibrium and necessarily reduces the role of higher order beliefs.

The remainder of this paper is as follows: section 2 discusses some contributions related to our research. Section 3 introduces the theoretical set-up and equilibrium solutions, while

[3]See Section 2 for a fuller review of the related literature on higher order beliefs.

Section 4 describes data implemented in the structural model. Section 5 presents the empirical estimation of the structural model, while some comments are proposed on Section 6 about the posterior estimates and policy implications. Section 7 concludes.

## 2. Related literature

Our theoretical framework follows in the tradition the literature on social value of information firstly popularized by Morris and Shin (2002) and afterwards analyzed by several contributions, see among others Colombo et al. (2014) and Myatt and Wallace (2014, 2015). The game has a similar structure to that found in the literature concerned with information sharing (e.g., Vives, 1997; Angeletos et al., 2004, 2007), where the Gaussian-quadratic model and the linear solutions are common hypothesis. This choice represents a simple and flexible tool to link the theoretical part to the empirical structural model. Such close relationship allows to get estimates of the parameters of the individual utility function through a reduced form parametrization. The combination between private and public information in coordinating settings was investigated by Hellwig and Veldkamp (2009) who show that adopting strategic complementarities in actions may incentivize the role that higher order beliefs play in determining exchange rates. We validate this point observing a reassessment of information. On one side we show that higher order beliefs play a large role when *naive* agents manifest individual preferences with a *fundamental* and a *coordination* motive without consistently evaluating macroeconomic fundamental. On the other side, results are relatively different when more *sophisticated* agents realize their forecasting rule is not sufficient to correctly perceive the exchange rate and use aggregate information at macro level.

In the last decade, equilibrium strategy *à la* Morris and Shin was further proposed and enriched in many settings with asymmetric information including financial markets (Allen et al., 2006), business cycle models (Angeletos and La'O, 2009), oligopolistic competition (Myatt and Wallace, 2015). The literature slightly distinguished between public and private information by looking at two signals. We intentionally make the same choice to test our structural model for the sake of simplicity. Of course multiple information sources may enrich the design and constitute a fertile ground for follow-up research.[4] Alternative studies that theoretically verified the effects of multiple information sources are among others, Angeletos et al. (2004, 2007), who consider an investment game evaluating welfare as aggregation of agents' outcomes.[5]

Our investigation also refers to a strand of macroeconomic theory modeling the behaviour of exchange rates.[6] The reason to study exchange rate market as an application of our model of higher order beliefs relies on the possibility to measure the impact that information at the individual and aggregate level may play in the agents' optimization process. Systematic biases in individual expectations away from the truth have been empirically tested since Lucas (1972). Considerable debate has been prompted to comprehend the limitations faced by agents in the acquisition process of information, see Sims (2003) and Woodford (2002), amongst others. We here propose a learning process of investors about the current exchange rate determining one-step-ahead expectations through Kalman filter (Coibion and Gorodnichenko, 2012). Bacchetta and Van Wincoop (2004) instead study how exchange rate may vary for reasons that are not influenced by observed macro fundamentals and elaborate the *scapegoat* theory. This theory suggests that when a change of exchange rate is observed, the market may attribute some shifts to observed macro indicators as natural scapegoats influencing trading strategies. Fratzscher et al. (2015) develop an empirical test of this theory using as a proxy of scapegoat fundamentals, Consensus Economics of London surveys of predictors. The authors find that the inclusion of these expectations improves the explaining power of the fundamentals.[7] Bacchetta and Van Wincoop (2006) focus instead on the order flow providing a possible explanation for the empirical results verified by Evans and Lyons (2002), Payne (2003), and Froot and Ramadorai (2005). Assuming that agents are risk averse, the authors show that due to the imperfect correlated signals among investors,

[4]See Melosi (2017) for an investigation of the signaling effects of monetary policy.

[5]A similar information structure is observed in Dewan and Myatt (2008) for political leadership or in a Lucas-Phelps island setting in Myatt and Wallace (2014).

[6]See Rossi (2013) for further details.

[7]In the short run the heterogeneity among investors' expectations may lead to overrate the random macroeconomic fundamental.

transitory shocks may continuously influence the dynamics of exchange rate. This results is confirmed by Bacchetta and Van Wincoop (2013) in the scapegoat view. They endogenously derive large time variation between exchange rates and fundamentals demonstrating that expectations of structural parameters are essential in the process. The main motivation is that agents are not able to perfectly distinguish fundamentals and structural parameters due to the uncertainty of information. Even when the structural parameters are constant, the expectation of these parameters is the main cause for the instability of the relation since it can vary significantly over time. This is true also in case of bayesian rationality.

The potential advantage of our framework compared to the previous ones is to propose a transparent learning structure in the determination of agents' optimal choice. The optimization process quantifies the role of both private and public information and is able to measure the effect of higher order beliefs in the system. Moreover such flexibility allows to propose two different models with and without consistency property according to the information of the market. Empirical results show that higher order beliefs mainly form agents' expectations and this is true even in case of consistency regime. Public source is as a coordinating device among investors which generate the process of expectations in accordance with the aggregations of their signals. From a macro perspective, our structural model describes accurate predictions both in the short and long run through a cointegration relationship between exchange rates and fundamentals.

The following section proposes a detailed analysis of the learning process and the formation of expectations in the market.

## 3. The model

### *3.1. Agents' learning process*

Suppose an economy populated by a finite series of $n$ agents. In period $t$, each trader $i$ observes a combination of noisy private and public signals about the exchange rate $s_t$, described by a random walk,

$$s_t = s_{t-1} + \gamma_t, \qquad \gamma_t \sim \mathcal{N}(0, \sigma_\gamma^2) \tag{1}$$

where $\gamma_t$ is a gaussian shock with mean zero, variance $\sigma_\gamma^2$, and precision $\rho_s \equiv \sigma_\gamma^{-2}$. Agents thus assign a priori beliefs to the exchange rate dynamics. This hypothesis is consistent with the findings of Meese and Rogoff (1983) and Engel and West (2005), amongst others. To predict exchange rates, we assume agents take advantage on receiving public and private signals. We summarize the combination of all public signals by describing a unique process, namely, the fundamental $f_t$, connected to the exchange rates as follows,

$$f_t = s_t + \eta_t, \qquad \eta_t \sim \mathcal{N}(0, \sigma_\eta^2) \tag{2}$$

whereas all individual private signals $x_{it}$ are modelled by,

$$x_{it} = s_t + \epsilon_{it}, \qquad \epsilon_{it} \sim \mathcal{N}(0, \sigma_\epsilon^2). \tag{3}$$

In particular, $\eta_t$ and each idiosyncratic noise $\epsilon_{it}$ are gaussian variables independent on $s_t$, with mean 0 and precisions respectively $\rho_f = 1/\sigma_\eta^2$ and $\rho_x = 1/\sigma_\epsilon^2$. So while information about the fundamental $f_t$ is common knowledge among agents, the private signal $x_{it}$ is specific to agent $i$ and not observed by other predictors. The set of latent variables $x_{it}$ describes the flow of private information agents use to improve their predictions. It is a set of aggregate time series that could in principle take into account different signals, even including order flow.[8] Individual posteriors beliefs about exchange rates are thus gaussian with average,

$$E_t^i[s_t|f_t; x_{it}] = \frac{\rho_{\tilde{y}}\tilde{y}_t + \rho_x x_{it}}{\rho_{\tilde{y}} + \rho_x} \tag{4}$$

[8] Unfortunately the order flow data are not available and we solve the problem through a latent process. We make inference on that process by using subjective expectations of Consensus Forecasts in the empirical analysis.

and precision $\rho[s_t|f_t;x_{it}] = \rho_{\tilde{y}} + \rho_x$. In particular, $\rho_{\tilde{y}} = \rho_s + \rho_f$, whereas $\tilde{y}_t = (\rho_s s_{t-1} + \rho_f f_t)/\rho_{\tilde{y}}$. The weight of the public signal in the Bayesian projection $s$ on the information set $H_i(t) = \{f_t; x_{it}\}$ is $\alpha_{\tilde{y}} = \rho_{\tilde{y}}/\left(\rho_{\tilde{y}} + \rho_x\right)$, while the weight of the private signal is $\alpha_{x_i} = \rho_x/\left(\rho_{\tilde{y}} + \rho_x\right)$. The posterior mean for each agent $i$ is then derived, i.e., $\mathbb{E}_t^i[s_t|f_t;x_{it}] = \alpha_{x_i}x_{it} + \alpha_{\tilde{y}}\tilde{y}_t$.

Following Hommes et al. (2005) and Branch and Evans (2011) amongst others, we assume that, when agents compute their forecasts, do not observe contemporaneous exchange rates $s_t$. There are three reasons justifying this assumption in our setup. First, we do not know *ex-ante* the exact instant when agents form their expectations. As stressed in Section 4, we know from survey data agents are asked to forecast spot rates at the second Monday of each month, and thus we consider as a benchmark rate, the exchange rate $s_t$ of that day. For instance, European Central Bank usually updates reference rates at *16:00* CET on every working day.[9] However agents, or at least some of them, could form their expectations with a different timing, or by using different market prices with respect to the Central Bank. We thus assume common information all agents observe at $t$ is $s_{t-1}$. Second, as suggested in Hommes et al. (2005), in models with heterogenous agents, asset prices realized at $t$ depend on aggregate expectations of future prices. Since agents at $t$ do not observe other agents' predictions, then they are unable to precisely evaluate the current price $s_t$, and therefore they just exploit information up to time $t-1$. Finally, some difficulties observing a unique price $s_t$ at time $t$ may arise because of the particular market microstructure of exchange rates. As stressed in Vitale (2007) and Menkhoff et al. (2016) amongst others, *FX* market is a decentralized, organized as an over-the-counter market where there is not necessarily a common observable price at any point in time.[10]

[9] Furthermore, the determination of *official* exchange rate by a Central Bank is a serious tangled procedure that requires several steps based on time, trading size and volatility. According to the methodology elaborated by European Central Bank, 'a detailed guideline for evaluating the real exchange rate is required in the belief that transparency in governance and the setting methodology is in the public interest and reinforces the credibility of the relevant reference rate'.

[10] To mimic these features in our model, in AppendixA, we propose an alternative characterization by explicitly taking into account that exchange rates are observed up to a measurement error. In particular, we model the observable exchange rate as a combination of the true (permanent) rate plus a measurement

Based on public and private signals at $t$, the expected prediction for $s_{t+1}$ is obtained through the recursion in eq. (1), i.e. $e_{it} = \mathbb{E}_t^i[s_{t+1}] = \mathbb{E}_t^i[s_t|x_{it}; f_t]$. The combined effect of personal and market expectations can be represented by the model of Morris and Shin (2002) which describes a version of Keynes' beauty contest. They propose a coordination game where traders' expectations have to meet both a fundamental and coordination motive: their expectations on exchange rates are close to the one-step-ahead rate and to the average value set by the market. The individual expected utility is the negative of a quadratic loss function with two components. The first component (the fundamental motive) is a standard quadratic loss increasing in the distance between individual expectation $e_{it}$ and the one-step-ahead spot $s_{t+1}$, while the second component (the coordination motive) is the *beauty contest* term, i.e., the loss is increasing in the distance between individual expectation $e_{it}$ and the average action of all others $\bar{e}_t = \frac{1}{n}\sum\limits_{j \neq i} e_{jt}$, that is, the average (or consensus) expected forecast of the other investors.[11] Explicitly, the utility function of each investor is given by:

$$U(e_{it}, \bar{e}_t, \sigma_e^2, s_{t+1}) = -(1-\delta)(e_{it} - s_{t+1})^2 - \delta(e_{it} - \bar{e}_t)^2 \tag{5}$$

where the parameter $\delta \in (0, 1)$ is a scalar related to the presence of higher order beliefs and describes the intensity of the coordination motive, i.e., the importance that agent $i$ attaches to the expectations of other market predictors. Under perfect information about the exchange rate $s_{t+1}$, due to symmetry ($e_{it}(.) = \bar{e}_t(.) = s_{t+1}$, $\forall i$), the best response is given by the unique equilibrium characteristics where the predictors' choice exactly coincides with their expectation. In case of imperfect information, in contrast, optimality is required for any $(x_{it}; f_t)$ in the predictor's choice. The quadratic specification of the utility function ensures the linearity of the predictors' best responses and efficient allocations. We here assume that the best prediction of each agent is based on current information, i.e., $x_{it}$ and $f_t$, even though

(transitory) error. We show that the dynamics of exchange rates is identical and the results are robust to the case proposed in this subsection.

[11] There are different applications of this framework in the beauty contest literature, see Morris and Shin (2002), Baeriswyl and Cornand (2010, 2014), Cornand and Heinemann (2008), Myatt and Wallace (2014, 2015).

a generalization that includes past information can be easily derived as suggested by Coibion and Gorodnichenko (2012).

According to the model proposed in this Section, it follows that,

**Proposition 1.** *Suppose the economy is described by eqs. (1), (2), (3) and $n$ agents minimize the loss function described in eq. (5). Assume also that each agent follows a linear strategy $e_{it}(x_{it}; f_t; \rho_{\tilde{y}}; \rho_x) = \varphi_x x_{it} + \varphi_{\tilde{y}} \tilde{y}_t$. Then the unique and optimal individual forecast is*

$$e_{it} = \mathbb{E}_t^i(s_{t+1}) = \varphi_x x_{it} + \varphi_{\tilde{y}} \tilde{y}_t, \tag{6}$$

*with optimal weights*

$$\varphi_x = \frac{(1-\varrho)\rho_x}{(1-\varrho)\rho_x + \rho_{\tilde{y}}} \qquad \textit{and} \qquad \varphi_{\tilde{y}} = \frac{\rho_{\tilde{y}}}{(1-\varrho)\rho_x + \rho_{\tilde{y}}} = 1 - \varphi_x, \tag{7}$$

*where in particular $\varrho = \frac{n\delta - \delta}{n - \delta}$.*

**Proof.** See AppendixB ■

Note that eq. (6) represents the *individual* perceived law of motion (*PLM*, hereafter) of each agent and describes how agents form expectations. It explains the individual ability to predict exchange rates. We may indeed observe that the sensitivity of the predictor's expectations to exchange rates is driven by two factors. First, the weight of the beauty contest factor, i.e., $\delta$ identifies the importance attached to the expectations of other predictors. Note that when $\delta = 0$, the predictor's optimal choice coincides with her personal expectation. Higher values of $\delta$ induce the agent mainly to take into account public sources of information when making his/her own prediction. Second, the sensitivity of the predictor's expectations to the exchange rate depends on the quality of private and public signals in terms of precision. Agents assign lower weights to private signal, while public source acts as a coordinating mechanism for prediction of others' action. Aggregating the individual predictions among $n$

investors from eq. (6) delivers

$$\bar{\mathbb{E}}_t[s_{t+1}] = \varphi_x \bar{x}_t + \varphi_{\tilde{y}} \tilde{y}_t \tag{8}$$

that is the average expectation, or *aggregate PLM* across all investors, where $\bar{x}_t = \sum_j x_{jt}/n$.

*3.2. Exchange rate dynamics*

Once the expectation formation scheme has been defined, we plug it into a standard macroeconomic setup. In particular we consider the exchange rate environment where prices are linked to private (Evans and Lyons, 2002) as well as to public information (Engel and West, 2005). Furthermore, survey data on expectations are available (Jongen et al., 2012; Fratzscher et al., 2015), allowing for empirical analysis and in particular to disentangle the impact of different sources of information.

Our aim is to derive the dynamics of the exchange rates by taking into account both the role of fundamentals and the agents' aggregate expectation previously obtained by the learning procedure. As standard in exchange rate models with heterogeneous rational investors, the uncovered interest rate parity condition implies market equilibrium,[12]

$$\bar{\mathbb{E}}_t[s_{t+1}] - s_t = i_t - i_t^* + \psi_t \tag{9}$$

where $i_t$ is the short term nominal interest rate, $i_t - i_t^*$ is the cross country interest rate differential, while $\psi_t$ is the liquidity premium. As stressed in Rossi (2013), there are many ways to describe this spread. Here, $(i_t - i_t^*)$ is modelled using a two-countries Taylor rule dynamics as in Lansing and Ma (2017) even though similar results can be obtained using different macroeconomic models.[13] We postulate that the home country nominal interest rate is determined by the central bank with the following Taylor rule,

$$i_t = \theta i_{t-1} + (1-\theta)\left[g_\pi \pi_t + g_y y_t + g_s(s_t - s_{t-1})\right] + \eta_t \tag{10}$$

[12] See AppendixC for a sketch of the derivation of eq. (9).

[13] In AppendixD, we propose an alternative standard model of exchange rate based on money and output differentials.

whereas for the foreign country is,

$$i_t^* = \theta i_{t-1}^* + (1-\theta)\left[g_\pi \pi_t^* + g_y y_t^*\right] + \eta_t^*. \tag{11}$$

where $\pi_t$ is the inflation rate (expressed in terms of the log difference of the price level), $y_t$ is the output gap, $\eta_t$ is an exogenous shock, the superscript $^*$ indicates foreign related variables, while $g_\pi$, $g_y$ and $g_s$, are parameters respectively of inflations, outputs and exchange rates, with $g_\pi > 1$ and $g_y, g_s > 0$. By plugging the interest rate differential into the uncovered interest rate parity condition of eq. (9), it follows that

$$s_t = \beta_1 f_t + \beta_2 \bar{x}_t + \beta_3 s_{t-1} + \beta_4 \psi_t + \varepsilon_{s,t} \tag{12}$$

where the quantity $f_t$ represents the macroeconomic fundamental, that is

$$f_t = \theta(i_{t-1} - i_{t-1}^*) + (1-\theta)g_\pi(\pi_t - \pi_t^*) + (1-\theta)g_y(y_t - y_t^*) \tag{13}$$

where $\beta_1 = (\varphi_{\tilde{y}}\tau_2 - 1)\lambda$, $\beta_2 = \lambda\varphi_x$, $\beta_3 = \varphi_{\tilde{y}}\tau_1\lambda + \lambda - 1$, $\beta_4 = -\lambda$ and $\varepsilon_{s,t} = \eta_t - \eta_t^*$. In particular, $\tau_1 = \frac{\rho_s}{\rho_{\tilde{y}}}$ and $\tau_2 = \frac{\rho_f}{\rho_{\tilde{y}}}$, $\lambda = \frac{1}{1+(1-\theta)g_s} \in (0,1)$ since $0 < \theta < 1$. This is the actual law of motion (*ALM*, henceforth) of the economy as the rule measuring the real value of the exchange rate in the system. The first term measures the contribution of the fundamental value in the determination of exchange rate, the second term captures the strategic interaction of higher order beliefs, the third term is the measure of persistency in the exchange rate, and the fourth term is the liquidity trade.

### *3.3. Consistent equilibria*

One of the main problems researchers face when dealing with heterogeneous expectations is to find conditions that make their forecasts about a relevant variable free from forecasting errors. The main purpose of these conditions is to make the model internally consistent between its components. Here we characterize internal consistency between microeconomic and macroeconomic parts, respectively proposed in subsections 3.1 and 3.2.

Following Hommes and Sorger (1998) and Hommes and Zhu (2014), we adopt a consistency definition to link the perceived and actual laws of motion in terms of statistical matching.[14] The intuition is that investors fail to recognize all explanatory variables affecting exchange rate dynamics. In particular they do not know precisely the *ALM,* but apply a parsimonious forecasting rule to predict exchange rates, i.e., they use their individual *PLM.* In this case, independently by received information, each agent builds her optimal forecasting rule potentially assigning unreliable weights to public and private signals, and these errors might be systematic. The magnitude of this error depends on the market exchange rate determination described by *ALM.* Having a model with internal consistency, instead, requires that agents remain rational and act in a more *sophisticated* way. In particular, their beliefs must be statistically coherent with observational evidences. Informed rational (or nearly rational) agents thus take advantage from *ALM* by looking at market equilibrium data on exchange rates, and exploit these information to build their forecasts. A possible way to carry market information into the forecasting problem is to assume that forecasting rules (*PLM*) and market dynamics (*ALM*) share at least some identical empirical moments. For instance, Hommes and Zhu (2014) require that the mean and the first-order autocorrelation of the $AR(1)$ perceived model are identical to the corresponding moments of the actual one. Similarly, Lansing and Ma (2017) use the moments induced by the *ALM* dynamics to derive OLS estimators for the regression parameters identifying the aggregate *PLM.*[15]

In our setup, by rearranging the aggregate PLM in eq. (8) gives that average forecasts can be computed from the following statistical model,

$$s_t = as_{t-1} + bf_t + (1 - a - b)\bar{x}_t + \delta_t \tag{14}$$

where $\{\delta_t\}$ is a white noise process, while $a = \varphi_{\tilde{y}}\tau_1$ and $b = \varphi_{\tilde{y}}\tau_2$ are parameters that can be estimated through OLS. On the other side, *ALM* is described by eq. (12).

[14] See also an application in Lansing (2010) and Lansing and Ma (2017).

[15] In particular, Lansing and Ma (2017) compute the parameter of OLS estimate as a covariance between exchange rate differential and the fundamental news from the observable data.

Our consistent procedure is therefore divided into three main steps as follows:

1. Identify $(a, b)$ from the moments of the aggregate $PLM$, i.e.,

$$a = \frac{Var\left(f_t - \bar{x}_t\right) cov\left(s_t - \bar{x}_t, s_{t-1} - \bar{x}_t\right) - cov\left(s_t - \bar{x}_t, f_t - \bar{x}_t\right) cov\left(s_{t-1} - \bar{x}_t, f_t - \bar{x}_t\right)}{Var\left(f_t - \bar{x}_t\right) Var\left(s_t - \bar{x}_t\right) - cov^2\left(s_t - \bar{x}_t, f_t - \bar{x}_t\right)} \tag{15}$$

$$b = \frac{Var\left(s_{t-1} - \bar{x}_t\right) cov\left(s_t - \bar{x}_t, f_t - \bar{x}_t\right) - cov\left(s_t - \bar{x}_t, s_{t-1} - \bar{x}_t\right) cov\left(s_{t-1} - \bar{x}_t, f_t - \bar{x}_t\right)}{Var\left(f_t - \bar{x}_t\right) Var\left(s_{t-1} - \bar{x}_t\right) - cov^2\left(s_{t-1} - \bar{x}_t, f_t - \bar{x}_t\right)} \tag{16}$$

2. Build an estimator for $(a, b)$ by using the theoretical moments induced by eq. (12), i.e., variances and covariances between $s_t$, $f_t$ and $\bar{x}_t$. Assuming eqs. (1) to (3) are valid even for $ALM$ implies there are some cointegration relationships connecting $s_t$, $f_t$ and $\bar{x}_t$. Plug these moments in eqs. (15-16) to get

$$\hat{a} = \frac{\lambda\rho_s + (1 - \rho_s(1 - \varphi_{\tilde{y}}) + \varphi_{\tilde{y}} + n\rho_f\varphi_{\tilde{y}}}{\rho_s(2\lambda + \rho_f + \rho_s) + n(\lambda\rho_f + \rho_s)\rho_x} \tag{17}$$

$$\hat{b} = \frac{\lambda(2\rho_s\rho_f + n\rho_s\rho_x) + \lambda((\rho_f^2 - \rho_f\rho_s)\tau_1 - \rho_s\rho_f - n\rho_s\rho_x - \rho_f^2)\varphi_{\tilde{y}}}{\rho_f(\rho_f + \rho_s + n\rho_x)} \tag{18}$$

3. Determine $\rho_s$ and $\rho_f$ such that $a = \varphi_{\tilde{y}}\tau_1 = \hat{a}$ and $b = \varphi_{\tilde{y}}\tau_2 = \hat{b}$.

All details of these computations are reported in AppendixE. In this way, estimators for the parameters of the $PLM$ are closely linked to the moments induced by the $ALM$. This matching thus corresponds to some constraints we need to impose on the parameter space of the expectations' formation model. Our definition of consistent equilibrium reads as follows:

**Definition 1.** *A consistent equilibrium is composed by individual and aggregate PLM, (6) and (8), by ALM (12) and by the subjective forecasting parameters* $(a, b)$ *satisfying,*

$$\varphi_{\tilde{y}}\tau_1 = \frac{\lambda\rho_s + (1 - \rho_s(1 - \varphi_{\tilde{y}}) + \varphi_{\tilde{y}} + n\rho_f\varphi_{\tilde{y}}}{\rho_s(2\lambda + \rho_f + \rho_s) + n(\lambda\rho_f + \rho_s)\rho_x} \tag{19}$$

$$\varphi_{\tilde{y}}\tau_2 = \frac{\lambda(2\rho_s\rho_f + n\rho_s\rho_x) + \lambda((\rho_f^2 - \rho_f\rho_s)\tau_1 - \rho_s\rho_f - n\rho_s\rho_x - \rho_f^2)\varphi_{\tilde{y}}}{\rho_f(\rho_f + \rho_s + n\rho_x)} \tag{20}$$

Finally, solving the recursive system of (19) and (20), we thereby reach the subsequent result:

**Proposition 2.** *A consistent equilibrium exists when the precisions, $\rho_s$ and $\rho_f$, satisfy:*

$$\rho_s = \frac{(n-\delta)\rho_f^2(\rho_f + \lambda\rho_f + n\rho_x)}{(1-\delta)(2n\lambda\rho_f\rho_x + n^2\lambda\rho_x^2 - (n-\delta)\rho_f^2)} \tag{21}$$

$$\rho_f = \frac{\delta(2\lambda - \rho_s) + n((\rho_s - 2) - \lambda\delta + \lambda(\rho_s - \rho_x)(\delta - \lambda)) + \Phi}{2n(n-\delta)\lambda} \tag{22}$$

*where*

$$\begin{aligned} \Phi \quad = \quad & ((n-\delta)(4n\lambda((n-\delta)\rho_s^2 + n((n+\lambda-\delta(1+\lambda)\rho_s)\rho_x \\ & -(1-\delta)\lambda + (n-\delta)(\rho_s - \lambda(2 - n(\rho_x - \rho_s)))^2))^{1/2} \end{aligned} \tag{23}$$

**Proof.** See AppendixF ■

Solving the system proposed in Definition 1, Proposition 2 allows to express precisions $\rho_s$ and $\rho_f$ as functions of other parameters. It is worth noting that, apart for some special cases, e.g., when $n \to \infty$, it is not possible to analytically solve the system.[16] However, numerical solutions are feasible.

In the next subsections, we compare the model under consistency with the unrestricted model without consistency. Under consistency, agents have an additional source of information, the market equilibrium, that allows for a more precise perception of economic signals. The reason is that in a model without consistency, *naive* agents choose their actions according

[16]See AppendixG where we evaluate these moments with the consequent restrictions of parameters when the number of agents in the economic system approaches infinity.

to fundamental motive (their own personal choice) and higher order beliefs (the average of others) without taking into account the role played by fundamentals. Instead, in a internally consistent model, agents still choose their actions based on the two components (fundamental motive and higher order beliefs) but perceive that the uncertain state of the economy is relatively more complex. Agents are therefore *sophisticated* and evaluate their choices within an integrated system which includes the evaluation of market equilibrium. This may determine a net change on the role that higher order beliefs play on agents' optimal choice.

## 4. Foreign exchange Consensus Survey data

We consider data on expectations obtained from the Foreign Exchange Consensus Forecasts ($FECF$) survey by *Consensus Economic of London*.

In this survey at the second Monday of each month, panelists are asked to forecast spot rates at different maturities. The sample is composed by almost 250 panelists spread all over the world and 40 of them are personally identifiable with their names. Some panelists provide systematically predictions in each publication, while some others appear with a lower frequency. There are also cases in which the panelist is included in the list although its prediction is not indicated.

Albeit the survey refers to different currencies, we focus on *euro/usd* and *usd/yen* one-month-ahead forecasts consisting of 78 observations per agent from January 2006 to June 2012. We also consider the *pound/usd* exchange rates since we collect expectations on a two-monthly frequency at the same time-span.

Our analysis is conducted by taking into account individual forecasts, i.e., forecasts reported by personally identifiable panel members in the publication. The average expectation of all members is also reported and indicated by *consensus forecast*.

For the *euro/usd* exchange rate, we build our dataset as follows. First we collect all individual forecasts from 2006 to 2012 and we record only forecasts from panelists with a response rate higher than 40 percent. We select 5 institutions, i.e., 15 time series of expectations. Note that 9 of them have just from 0 to 5 missing data and only in one case we noticed a response rate smaller than 50 percent. On average, the response rate for the US$ vs Euro one-month-ahead forecasts is around 90%.

It is worth noting that we observe for each month at least 11 individual expectations, while the average number of respondents per month is larger than 13, i.e., a value sufficient for cross sectional heterogeneity among forecasters. Figure 1 evidences that average one-month-ahead forecasts approximate fairly well actual exchange rates.

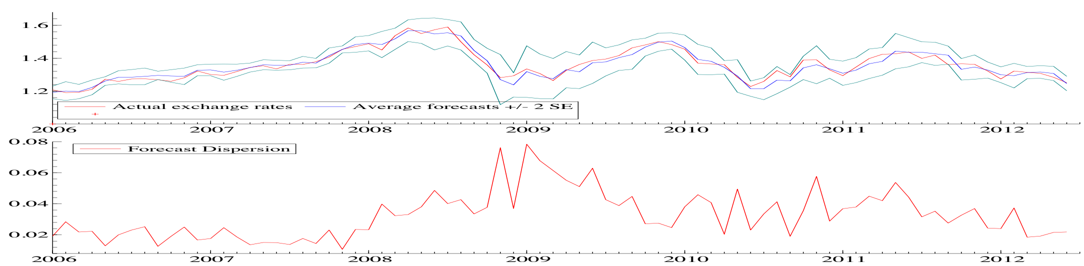


Figure 1: **Euro/Usd.** Upper panel: Actual vs average forecasts together with a 95% confidence interval from January 2006 to June 2012. Lower panel: Estimated dispersion over time.

As stressed by Jongen et al. (2012), expectations are dispersed confirming heterogeneity among panelists. In particular, the lower panel of Figure 1 describes that dispersion is relatively moderate from January 2006 to September 2007, then it increases until reaching a peak in January 2009 and finally declines and stabilizes from November 2009 to June 2012.

Regarding the representativeness of panelists, it is worth noting that some of them represent major dealing banks, whose names are reported in Table 1. It includes also the actual sample size and their relevance in term of market size.

In particular, 8 out of 15 institutions are included in the Top 10 currency traders of the Forex market and represent about 70 percent of market share of exchange rates. Furthermore, some of these institutions provide trading platforms to smaller banks. This procedure is called *white labeling* and is highly efficient for market dynamics, although it induces a concentration of information. In fact, large banks can observe directly small banks' trading flows and extract from these data possibly relevant information at lower costs (King et al., 2012).

For *pound/usd* and *yen/usd*, we consider the following list of forecasters, i.e., ABN Amro,

Table 1: **Euro/Usd.** Predictions of individual forecasters: missing observations and percentage on the total number of observations in the sample. Last two columns: companies with proprietary trading platforms and if they are in top 10 currency traders list.

| | n missing | % on total | white label | Top 10 |
|---|---|---|---|---|
| Bank of Tokio - Mitsubushi | 1 | 1 | | |
| Barclays Capital | 31 | 40 | yes | yes |
| BNP Paribas | 7 | 9 | | yes |
| BoA - Merril Linch | 4 | 5 | | yes |
| Citigroup | 45 | 57 | yes | yes |
| Commerzbank | 4 | 5 | | |
| Deutsche Bank Research | 18 | 23 | yes | yes |
| General Motors | 0 | 0 | | |
| HSBC | 1 | 1 | yes | yes |
| IHS Global Insight | 0 | 0 | | |
| J.P. Morgan | 10 | 12 | yes | yes |
| Oxford Economics | 7 | 9 | | |
| Royal Bank of Canada | 0 | 0 | | |
| UBS | 3 | 4 | yes | yes |
| WestLB | 2 | 4 | | |

Bank of Tokio - Mitsubushi, BNP Paribas, Bank of America - Merril Linch, Commerzbank, General Motors, HSBC, Global Insight, ING Financial Markets, JPMorgan, Oxford Economics, Royal Bank of Canada, Société Générale, UBS and WestLB. We obtained individual forecasts only on a two-monthly frequency for *pound/usd*. However, the average number of respondents per month is about 14 when the information is available. Figure 2 displays average forecasts compared to actual exchange rates and the dispersion on expectations and evidence that the larger level of heterogeneity is observed in 2009.

Finally, for the *usd/yen* case the response rate is larger than 95%. In particular, the average number of forecasts for each months is larger than 14 and just for two panelists the response rate is smaller than 90% Figure 3 shows actual data, average forecasts and dispersions. In this case the level of dispersion is substantially in line with what observed for the other two exchange rates even though the peak observed in 2009 is less evident.

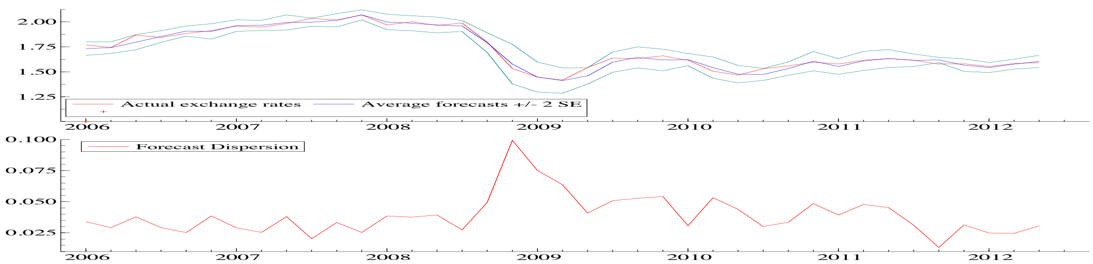


Figure 2: **Pound/Usd**. Upper panel: Actual vs average forecasts together with a 95% confidence interval from January 2006 to June 2012. Lower panel: Estimated dispersion over time

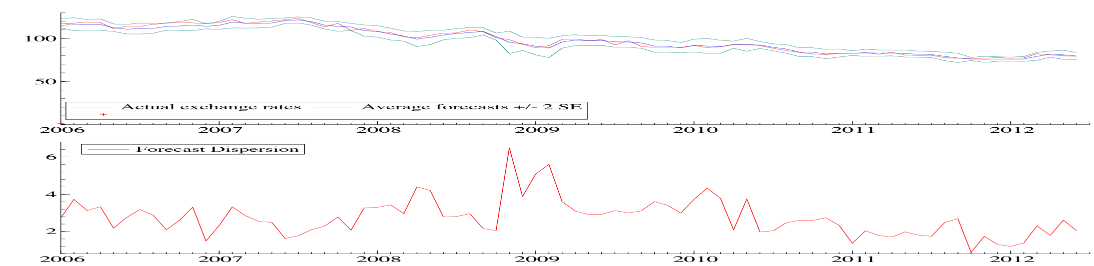


Figure 3: **Usd/Yen**. Upper panel: Actual vs average forecasts together with a 95% confidence interval from January 2006 to June 2012. Lower panel: Estimated dispersion over time

## 5. Empirical Model

### *5.1. Methods and Data*

We consider a state-space model to describe the economic environment as a combination of learning mechanism and the equilibrium model for exchange rates as proposed in Section 3. Our empirical strategy is developed to closely replicate the theoretical framework. It can be summarized by the following equations,

$$
\begin{aligned}
s_t &= \beta_1 f_t + \beta_2 \bar{x}_t + \beta_3 s_{t-1} + \beta_4 \psi_t + \epsilon_{s,t} && (24)\\
f_t &= f_{t-1} + \theta_f \epsilon_{f,t-1} + \epsilon_{f,t} && (25)\\
\tilde{y}_t &= \frac{\rho_f}{\rho_f + \rho_s} f_t + \left(1 - \frac{\rho_f}{\rho_f + \rho_s}\right) s_{t-1} && (26)\\
\psi_t &= \rho_\psi \psi_{t-1} + \epsilon_{\psi,t} && (27)\\
x_{it} &= x_{it-1} + \theta_x \epsilon_{x_i,t-1} + \epsilon_{x_i,t}, \qquad i = 1, \ldots, N && (28)
\end{aligned}
$$

Equation (24) is the empirical counterpart of eq. (12) that describes in equilibrium the dynamics of exchange rates as a function of (aggregate) private information, $\bar{x}_t$, economic fundamentals, $f_t$, and past exchange rate, $s_{t-1}$. The coefficients $\beta_i$ are explicit functions of the structural parameters $\rho_f$, $\rho_s$, $\rho_x$, $\delta$ and $\lambda$.

Equation (25) is the link between exchange rates and fundamentals as defined in eq. (2), i.e., $f_t = s_t + \eta_t$. Following Harvey (1990), it is in fact easy to prove that eq. (2), together with the random walk hypothesis for exchange rates of eq. (1), corresponds to a standard ARIMA$(0,1,1)$ process with MA parameter $\theta_f = \left(\sqrt{q_f^2 + 4q_f} - 2 - q_f\right)/2$ and gaussian noise, $\epsilon_{f,t}$, with variance $\sigma_f^2 = -\sigma_\eta^2/\theta_f$, where $q_f = \sigma_\gamma^2/\sigma_\eta^2$ is the signal-noise ratio. The non stationarity of the fundamental process is consistent for instance with Bacchetta and Van Wincoop (2006). In our setup, this feature is reasonable due to the non-stationary nature of the exchange rates, their expectations and most of their determinants (Engel and West, 2005). Similarly, eq. (28) corresponds to the definition of the private signal $x_{it} = s_t + \epsilon_{it}$ defined in eqs. (1) and (3).

Eq. (26) represents the dynamics of public information defined as a convex combination of past exchange rates and fundamentals as derived in Section 3, while $\psi_t$ is an autoregressive process and in our empirical exercise we set $\rho_\psi = 1$.

The shocks $\epsilon_t = (\epsilon_{s,t}, \epsilon_{f,t}, \epsilon_{\psi,t}, \epsilon_{x_i,t})$, $i = 1, \ldots, N$ are all Gaussian with mean zero and standard deviation, respectively $\sigma_s, \sigma_f, \sigma_\psi$ and $\sigma_{x_i}$, whereas $N$ is the number of informed agents that make predictions on exchange rates. The model defined in eqs. (24-28) can be rewritten in compact form as

$$\Gamma_0 \mathbf{x}_t = c_x + \Gamma_1 \mathbf{x}_{t-1} + \Gamma_\epsilon \boldsymbol{\epsilon}_t \tag{29}$$

and in particular $\mathbf{x}_t = (s_t, f_t, \tilde{y}_t, \psi_t, x_{it})$, $i = 1, \dots, N$, while $\Gamma_0, \Gamma_1$ and $\Gamma_\epsilon$ are appropriate square matrices of parameters. By pre-multiplying eq. (29) with $\Gamma_0^{-1}$ we get

$$\mathbf{x}_t = \Theta_c + \Theta_x \mathbf{x}_{t-1} + \Theta_\epsilon \boldsymbol{\epsilon}_t. \tag{30}$$

Note that these dynamics refer to potentially non observable variables, and for this reason they define the transition equation of a gaussian state-space model. The model is then linked to a set of observables by defining the measurement equations. For our empirical analysis, we consider as observables the current exchange rates, $s_t$, the expectations $\mathbb{E}_t^i[s_{t+1}]$ which are represented by our dataset on heterogeneous survey forecasts concerning the actual exchange rates and three fundamentals $f_{i,t}$, $i = 1, 2, 3$. Measurement equations are thus,

$$\mathbb{E}_t^i[s_{t+1}] = \varphi_{\tilde{y}} \tilde{y}_t + \varphi_x x_{it} + \epsilon_{e_i,t}, \qquad \epsilon_{e_i,t} \sim \mathcal{N}(0, \sigma_{E_i}^2) \qquad i = 1, \dots, N \tag{31}$$

$$f_{j,t} = \alpha_j f_t + \epsilon_{j,t}, \qquad \epsilon_{j,t} \sim \mathcal{N}(0, \sigma_{f_j}^2) \qquad j = 1, \dots, 3 \tag{32}$$

$$\hat{s}_t = s_t \tag{33}$$

that in compact form reads as

$$\hat{\boldsymbol{y}}_t = S \mathbf{x}_t + \boldsymbol{\epsilon}_{\boldsymbol{y},\boldsymbol{t}}, \tag{34}$$

where the observables are $\hat{\boldsymbol{y}}_t = \left(\mathbb{E}_t^i[s_{t+1}], f_{j,t}, \hat{s}_t\right)$, $i = 1, \dots, N$, $j = 1, \dots, 3$, $S$ is a matrix of coefficients and $\boldsymbol{\epsilon}_{\boldsymbol{y},\boldsymbol{t}}$ is the vector of measurement errors. In particular we use the symbol $\hat{}$ to distinguish between observed and theoretical variables.

Eq. (33) defines the connection between the logarithmic equilibrium exchange rate provided by our macroeconomic model and the observed one. In turn, eq. (31) identifies the mechanism that forms individual expectations as a mixed effect of private and public information weighted respectively by $\varphi_x$ and $\varphi_{\tilde{y}}$ as in eq. (7).

It is worth noting that $f_t$ is a linear combination of macroeconomic indicators as shown in eq. (13). Such indicators allow to measure the public signal $f_t$ summarizing the economic dynamics in the market. To define a link between a potential non observed factor and some data from eq. (13) and eq. (25), we apply the dynamic factor model (see Forni et al., 2000; Stock and Watson, 2011 for a survey on these methods). A similar approach has been proposed in Boivin and Giannoni (2006) for *DSGE* models estimation. In particular, $f_t$ is measured through a set of observables that are actually driven by the factor itself. This strategy has the advantage that we do not need to explicitly describe the dynamics of each macroeconomic factor $f_{j,t}$, but just the aggregate one as a public signal, thus keeping our empirical model consistent with the theoretical framework. According to our Taylor rule definition, $f_{1,t} = \pi_t - \pi_t^*$, $f_{2,t} = y_t - y_t^*$ and $f_{3,t} = i_{t-1} - i_{t-1}^*$, are the differential of the inflation rates, the differential of output gap and past short term interest rates respectively for the home and the foreign countries. Finally, $\epsilon_{e_i,t}$, $\epsilon_{j,t}$ can be considered as measurement errors. To avoid some identification issues, we set the variances of $\epsilon_{j,t}$ to 1. We consider observable expectations $\mathbb{E}_t^i[s_{t+1}]$ for $N = 15$ institutions which represent the most influential companies providing predictions for exchange rates in the whole market as suggested in Section 4.

Data on macroeconomic fundamentals have been obtained from the Federal Reserve Economic Data (FRED) database. In particular we used monthly data on Consumer Price Indexes, short term nominal interest rates and Industrial Production Indexes. A measure of the output gap has been built by removing the trend from the logarithm of the Industrial Production index through the Hodrick-Prescott filter (see Lansing and Ma, 2017 for a similar strategy). Finally, exchange rates have been observed the *business* day before the survey was conducted as suggested by Fratzscher et al. (2015). It is worth noting that our database on subjective forecasts is affected by missing values. This is not a significant problem since the Kalman filter predicts missing data and allows for the computation of the likelihood function in a natural way (see Koopman et al., 1999 for a treatment on this point).

*5.2. Prior distributions and inferential methods*

To make inference on structural parameters, we recur to Bayesian estimation methods here, and in particular to Markov chain Monte Carlo algorithms[17] ($MCMC$) which have proved to be successful in the empirical macroeconomic literature (Kim and Pagan, 1995; Canova, 2007). In particular, as standard practice for DSGE models (An and Schorfheide, 2007), we update the structural parameters in block through a Random Walk Metropolis Hastings algorithm and then, for each draw, we compute likelihood and acceptance probabilities using the state-space representation of eqs. (30) and (34).

Our first interest is to capture the effect of higher order beliefs on the dynamics of the exchange rate. This is identified by the weight $\delta$ in the decision process of the individual predictor. Our second task is to measure the role of private and public information to determine actual expectations. We need to explore the coefficients $\varphi_x$ and $\varphi_{\tilde{y}}$ obtained from eq. (7).

In the theoretical model of Section 3, we show that the coefficient $\varphi_x$ measures the relevance of private information in forming expectations, while, $\varphi_{\tilde{y}}$ captures the relevance of public information. There is also an influence of the value of $\delta$ on the dimensions of $\varphi_x$ and $\varphi_{\tilde{y}}$. A higher $\delta$ reflects a larger weight to public signal with respect to the private one. Our prior choices on the transition equation parameters are summarized in Table 2, whereas for the measurement equation parameters are reported in Table 3.

Overall, we considered prior densities that match the domain of the structural parameters. In particular, we select a prior distribution for $\delta$ with average 0.5 (and standard deviation 0.025),[18] consequently assigning an equal weight to the two incentives in the decision-making function of our predictors (eq. 5). A priori, we assume that public and private information play the same role when agents form their own expectations, i.e., without forcing the model to privilege certain sources of information. This guess is consistent with the hypothesis that $\varphi_x$

[17]See Robert and Casella (1999, ch. 6-7) for a general treatment on MCMC algorithms and Monte Carlo methods in general.

[18]The results are robust with different values of standard deviation.

and $\varphi_{\tilde{y}}$ are equal. Since these weights depend on the precision coefficients $\rho_f$, $\rho_s$, $\rho_x$ and on $\delta$, we need to find prior distributions for them that at least on average, give $\mathbb{E}[\varphi_x] = \mathbb{E}[\varphi_{\tilde{y}}] = 0.5$.

We distinguish between two alternatives: first, the *consistency* case in which agents are boundedly rational and benefit from getting information from market equilibrium and second, the *non-consistency* case where agents form their expectations by relying just on the learning game. In the non-consistency analysis, we set the prior distributions for $\rho_f$ and $\rho_s$ as Gamma with mean 1 and standard deviation 0.1, whereas $\rho_x$ is still Gamma, but with larger expected value, namely, 4 and standard deviation 0.2.[19] Following Lansing and Ma (2017), we expect the discount factor $\lambda$ on average close to 1 and for this reason we set the prior distribution to a Beta variable with mean 0.95 and standard deviation 0.025. In the consistency analysis, instead the condition $\mathbb{E}[\varphi_x] = \mathbb{E}[\varphi_{\tilde{y}}] = 0.5$, is met if a priori the average of $\lambda$ is 0.67, the average of $\rho_x$ is 4 and the average of $\delta$ is 0.5. Of course, the average of $\rho_s$ and $\rho_x$ are functions of the other parameters by Proposition 2. Note that in Table 2, the priors of parameters, respectively $\beta_1$, $\beta_2$, $\beta_3$, $\beta_4$,$\theta_f$,$\sigma_f$,$\theta_s$,$\sigma_s$,$\theta_x$,$\sigma_x$, are not shown since they are functions of the other parameters as described in Section 3. The standard deviation are, respectively, 0.05, 0.2 and 0.025. We assume a rather informative prior for $\alpha_1$, $\alpha_2$ and $\alpha_3$ coherently with Lansing and Ma (2017) that are Gamma with mean 1.1, 3.0 and 0.5 and standard deviation 0.1. The standard deviations of the shocks, including standard deviations of the measurement errors are relatively dispersed. Their standard deviations in particular are quite large with respect to the corresponding expected values, i.e., Inverse Gamma variables with mean 0.3 and standard deviation 0.1.

## 6. Posterior estimates and policy implications

All computations are based on software written using the Ox© 7.0 language of Doornik (2001) combined with the state space library `ssfpack` of Koopman et al. (1999). Posterior estimates were obtained by running $150,000$ iterations of the $MCMC$ algorithm with a

[19] An extensive sensitivity analysis suggests that posterior estimates of $\varphi_x$ and $\varphi_{\tilde{y}}$ are robust with respect to this choice.

burn-in of $50,000$ which is a sufficient number of iterations to remove dependence on initial conditions. As standard practice in macro-econometrics, initial conditions were obtained by maximizing the posterior mode for the parameters.

Our empirical analysis is developed by systematically comparing the results in case of consistency and non-consistency as a test for the theoretical model in Section 3. The reference rate is the *eur/usd* currency and results are summarized in Table 2.

Table 2: **Euro/usd**: Posterior computation (MCMC) - Structural parameters. Note that the prior distributions are defined through their averages and variances in parenthesis.

| | Consistency | | | Non Consistency | | |
|---|---|---|---|---|---|---|
| | Mean | 95% Cred. Int. | Prior | Mean | 95% Cred. Int. | Prior |
| $\beta_1$ | -0.5410 | [-0.546, -0.537] | | -0.6885 | [-0.760, -0.598] | |
| $\beta_2$ | 0.4506 | [0.446, 0.455] | | 0.0460 | [0.037, 0.057] | |
| $\beta_3$ | 0.4535 | [0.449, 0.457] | | 0.7540 | [0.704, 0.800] | |
| $\beta_4$ | -0.9744 | [-0.983, -0.964] | | -0.8884 | [-0.960, -0.781] | |
| $\rho_f$ | 4.8073 | [4.318, 5.278] | | 1.0995 | [0.884, 1.378] | $\mathcal{G}(1, 0.1)$ |
| $\rho_s$ | 4.7456 | [4.257, 5.215] | | 3.5322 | [3.162, 3.912] | $\mathcal{G}(1, 0.1)$ |
| $\rho_x$ | 7.3588 | [6.746, 7.942] | $\mathcal{G}(4.0, 0.2)$ | 2.5928 | [2.299, 2.909] | $\mathcal{G}(4, 0.2)$ |
| $\lambda$ | 0.9744 | [0.964, 0.983] | $\mathcal{B}(0.67, 0.05)$ | 0.8884 | [0.781, 0.960] | $\mathcal{B}(0.95, 0.025)$ |
| $\delta$ | 0.3361 | [0.300, 0.374] | $\mathcal{B}(0.5, 0.025)$ | 0.9081 | [0.885, 0.926] | $\mathcal{B}(0.5, 0.025)$ |
| $\varphi_{\tilde{y}}$ | 0.5376 | [0.532, 0.543] | | 0.9482 | [0.937, 0.957] | |
| $\varphi_x$ | 0.4624 | [0.457, 0.468] | | 0.0518 | [0.043, 0.063] | |
| $\sigma_s$ | 0.2268 | [0.188, 0.277] | $\mathcal{IG}(0.6, 0.2)$ | 0.2121 | [0.177, 0.258] | $\mathcal{IG}(0.6, 0.2)$ |
| $\sigma_\phi$ | 0.2501 | [0.204, 0.306] | $\mathcal{IG}(0.6, 0.2)$ | 0.2361 | [0.194, 0.287] | $\mathcal{IG}(0.6, 0.2)$ |
| $\theta_f$ | -0.1504 | [-0.162, -0.140] | | -0.5769 | [-0.615, -0.534] | |
| $\sigma_f$ | 2.5796 | [2.481, 2.672] | | 1.2610 | [1.164, 1.361] | |
| $\theta_x$ | -0.1082 | [-0.116, -0.102] | | -0.4350 | [-0.461, -0.409] | |
| $\sigma_x$ | 1.1214 | [1.113, 1.131] | | 0.9429 | [0.903, 0.984] | |

The posterior computations for the other parameters as $\alpha_j, \forall\, j \in \{1, 2, 3\}$ of fundamentals and $\sigma$ values for 15 institutions are reported in Table 3.

For both consistency and non consistency, Table 2 includes posterior estimates of the structural relevant parameters of the transition equations, namely estimated posterior averages, the 95% credibility intervals for the parameters and the priors for the free paramaters as discussed in subsection 5.2. Figures 4 and 5 display prior versus posterior comparisons in

Table 3: **Euro/usd**: Posterior computation (MCMC) - Measurement equation parameters. Note that the prior distributions are defined through their averages and variances in parenthesis.

| | Consistency | | | Non Consistency | | |
|---|---|---|---|---|---|---|
| | Mean | 95% Cred. Int. | Prior | Mean | 95% Cred. Int. | Prior |
| $\alpha_1$ | 0.9696 | [0.809, 1.135] | $\mathcal{N}(1.1, 0.1)$ | 1.0623 | [0.880, 1.257] | $\mathcal{N}(1.1, 0.1)$ |
| $\alpha_2$ | 3.0464 | [2.860, 3.240] | $\mathcal{N}(3.0, 0.1)$ | 3.0754 | [2.894, 3.247] | $\mathcal{N}(3.0, 0.1)$ |
| $\alpha_3$ | 0.6845 | [0.452, 0.943] | $\mathcal{N}(0.5, 0.1)$ | 0.6128 | [0.393, 0.872] | $\mathcal{N}(0.5, 0.1)$ |
| $\sigma_{E1}$ | 0.1441 | [0.103, 0.201] | $\mathcal{IG}(0.3, 0.1)$ | 0.0924 | [0.082, 0.106] | $\mathcal{IG}(0.3, 0.1)$ |
| $\sigma_{E2}$ | 0.1513 | [0.120, 0.188] | $\mathcal{IG}(0.3, 0.1)$ | 0.1127 | [0.088, 0.142] | $\mathcal{IG}(0.3, 0.1)$ |
| $\sigma_{E3}$ | 0.1503 | [0.123, 0.189] | $\mathcal{IG}(0.3, 0.1)$ | 0.1114 | [0.092, 0.145] | $\mathcal{IG}(0.3, 0.1)$ |
| $\sigma_{E4}$ | 0.1502 | [0.113, 0.187] | $\mathcal{IG}(0.3, 0.1)$ | 0.1023 | [0.084, 0.121] | $\mathcal{IG}(0.3, 0.1)$ |
| $\sigma_{E5}$ | 0.3848 | [0.183, 0.855] | $\mathcal{IG}(0.3, 0.1)$ | 0.1579 | [0.125, 0.234] | $\mathcal{IG}(0.3, 0.1)$ |
| $\sigma_{E6}$ | 0.1283 | [0.104, 0.161] | $\mathcal{IG}(0.3, 0.1)$ | 0.0991 | [0.082, 0.124] | $\mathcal{IG}(0.3, 0.1)$ |
| $\sigma_{E7}$ | 0.1813 | [0.142, 0.218] | $\mathcal{IG}(0.3, 0.1)$ | 0.1023 | [0.089, 0.120] | $\mathcal{IG}(0.3, 0.1)$ |
| $\sigma_{E8}$ | 0.1432 | [0.115, 0.187] | $\mathcal{IG}(0.3, 0.1)$ | 0.0928 | [0.079, 0.109] | $\mathcal{IG}(0.3, 0.1)$ |
| $\sigma_{E9}$ | 0.1584 | [0.127, 0.232] | $\mathcal{IG}(0.3, 0.1)$ | 0.0961 | [0.084, 0.109] | $\mathcal{IG}(0.3, 0.1)$ |
| $\sigma_{E10}$ | 0.1640 | [0.131, 0.252] | $\mathcal{IG}(0.3, 0.1)$ | 0.0976 | [0.080, 0.129] | $\mathcal{IG}(0.3, 0.1)$ |
| $\sigma_{E11}$ | 0.1477 | [0.117, 0.189] | $\mathcal{IG}(0.3, 0.1)$ | 0.0978 | [0.082, 0.115] | $\mathcal{IG}(0.3, 0.1)$ |
| $\sigma_{E12}$ | 0.1253 | [0.104, 0.149] | $\mathcal{IG}(0.3, 0.1)$ | 0.0942 | [0.076, 0.109] | $\mathcal{IG}(0.3, 0.1)$ |
| $\sigma_{E13}$ | 0.1494 | [0.124, 0.190] | $\mathcal{IG}(0.3, 0.1)$ | 0.0956 | [0.080, 0.127] | $\mathcal{IG}(0.3, 0.1)$ |
| $\sigma_{E14}$ | 0.1432 | [0.110, 0.168] | $\mathcal{IG}(0.3, 0.1)$ | 0.1414 | [0.082, 0.337] | $\mathcal{IG}(0.3, 0.1)$ |
| $\sigma_{E15}$ | 0.1397 | [0.119, 0.176] | $\mathcal{IG}(0.3, 0.1)$ | 0.1021 | [0.081, 0.132] | $\mathcal{IG}(0.3, 0.1)$ |

case of consistency and non consistency for the free parameters $\rho_x$, $\lambda$ and $\delta$ for the consistency case and the free parameters $\rho_s, \rho_f, \rho_x$, $\lambda$ and $\delta$ for the non consistency case.

In the same figures, we show only the posteriors of the other parameters, respectively $\beta_1$, $\beta_2$, $\beta_3$, $\beta_4$, $\rho_s$,$\rho_f$,$\theta_f$,$\sigma_f$,$\theta_s$,$\sigma_s$,$\theta_x$,$\sigma_x$ for consistency, and $\beta_1$, $\beta_2$, $\beta_3$, $\beta_4$, $\theta_f$,$\sigma_f$,$\theta_s$,$\sigma_s$,$\theta_x$,$\sigma_x$ for non consistency, derived as a function of the free parameters. To validate our results, we also estimate the models using different exchange rates, i.e., *pound/usd* and *usd/yen*. Results for these datasets are reported in AppendixH.

It is interesting to note that prior and posterior distributions differ substantially from each other suggesting that the contribution of the data/likelihood is relevant and the relevance of the prior assumptions do not drive the posterior results.[20] This is true for all relevant variables in place in both consistent and non-consistent model.

We first comment the results about the non consistency model which corresponds to the

[20]This evidence is supported also by many robustness checks available upon request.

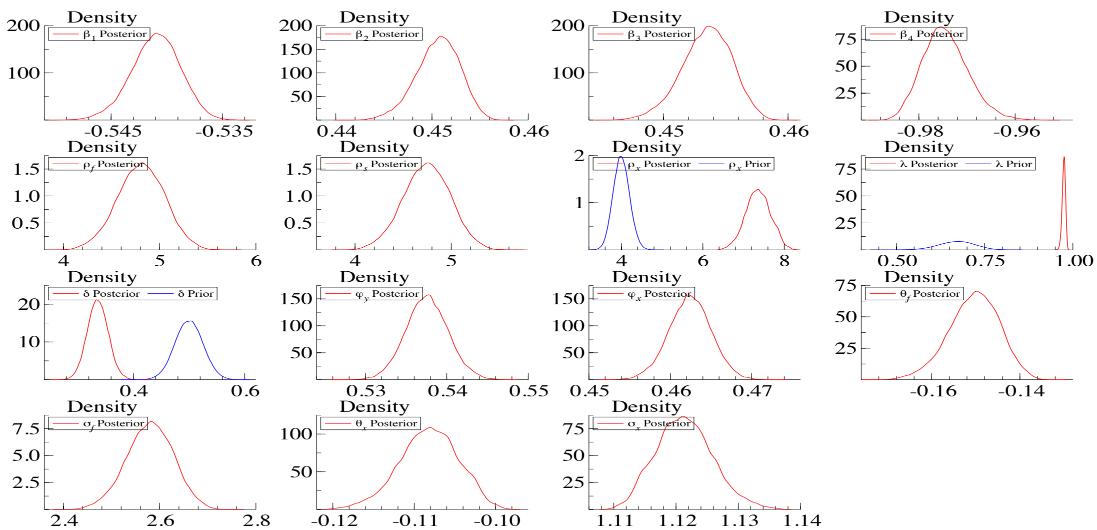

Figure 4: **Euro/usd**: Prior vs. Posterior distributions of the structural parameters. The *consistency* constraints on the precision of signals are imposed.

second part of Table 2. The first interesting result of our analysis refers to the parameter $\delta$ that is about 0.9. This is the measure of higher order beliefs and suggests that traders form their expectations by giving a large weight (90%) to the beliefs of the other agents. Looking at Figure 5, we observe a noticeable shift to the right of $\delta$ posterior in comparison of its prior distribution thus confirming the important role of the beauty contest mechanism in the predictor's evaluation process. The second important aspect is the role that public and private information, i.e., $\varphi_x$ and $\varphi_{\tilde{y}}$, play in individual forecasts. Table 2 indeed suggests that for non consistent case, the posterior average of public information $\varphi_{\tilde{y}}$ is almost 0.94, thus suggesting that public information accounts for about 94% of predictions, whereas just 6% are based on private information. This is coherent with the previous theoretical results related to the weights associated with higher order beliefs. The combination of higher order beliefs and information structure ensures relatively rational behavior in the decision-making process. When agents attach more importance to the consensus prediction than to their own personal assessment, they implicitly reduce the weight assigned to their private signal. This

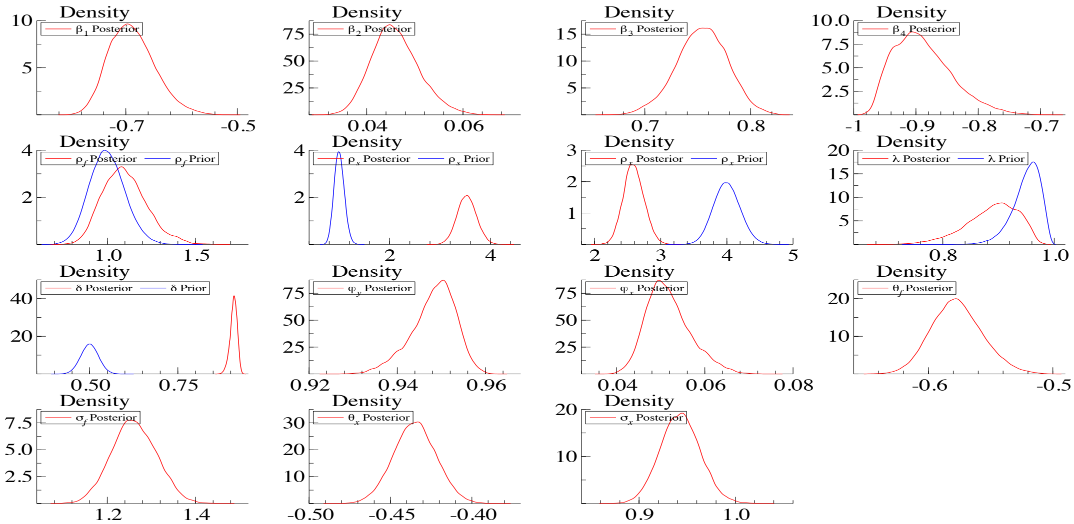


Figure 5: **Euro/usd**: Prior vs. Posterior distributions of the structural parameters. In this version $\rho_f$ and $\rho_s$ are free parameters

finding is consistent with different realizations of the posterior distributions compared to the prior ones for $\varphi_x$ and $\varphi_{\tilde{y}}$ as shown in Figure 5. As said in the previous section, our prior choices of parameters state that on average $\varphi_{\tilde{y}}$ and $\varphi_x$ have same expected value, i.e., 0.5. The role of higher order beliefs is thus largely confirmed even if the precision of the private signal, $\rho_x$, is more accurate than the precision of the public one, $\rho_f$, i.e., 2.59 vs 1.09 in Table 2.[21]

These results suggest that private information remains important in the determination of expectations and is necessary for a correct specification of the model. From an econometric point of view, excluding private information may induce mis-specification of the exchange rate dynamics. To confirm this finding, we also evaluated the short run prediction by using a naive rational expectation model with no private signals that closely mimics the dynamics

[21]Note that the precision of the exchange rate $\rho_s$ is higher than the other two (public and private) precisions, $\rho_x$ and $\rho_f$ for the non consistency case.

defined in this Section. In particular, we consider

$$s_t = \lambda \mathbb{E}_t[s_{t+1}] + (1 - \lambda) f_t - \lambda \psi_t + \epsilon_t, \tag{35}$$

in which $f_t$ is a combination of macroeconomic factors used in the Taylor rule, while $\psi_t$ is an i.i.d. sequence. Furthermore, rational expectations are defined such that $E_t[s_{t+1}] = s_t + \eta_t$, where $\eta_t$ is a Gaussian shock with mean zero and constant variance. We estimated this rational expectation model using MCMC. Specifically, for each posterior draw of the parameters, we solved the rational expectation system using Sims (2002) and implementing it with the Ox package LiRE developed by Mavroeidis and Zwols (2007). Then, for each parameter, we simulated the one-step-ahead prediction produced by the rational expectation model. An estimate of the rational expectations is provided in Figure 6.

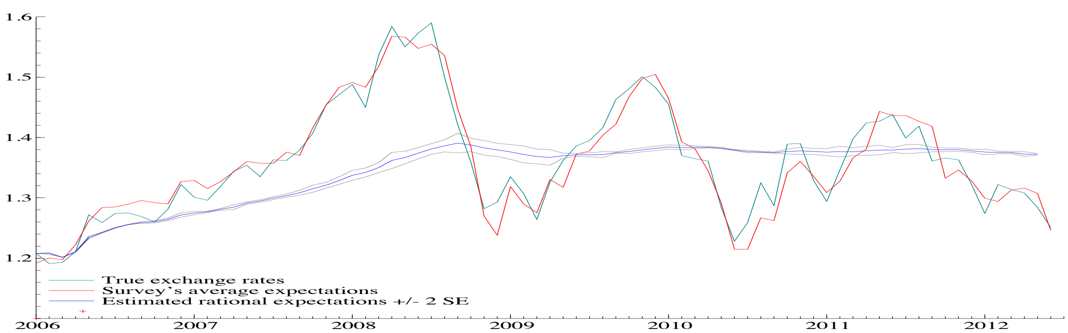


Figure 6: Actual exchange rates (green line), average from the survey (red line) together with the estimates of the expectations $E_t[s_{t+1}]$ (blue line)

It is worth noting that one-step-ahead predictions closely follows the estimates of rational expectations. It is thus evident that this standard rational expectation model poorly describe the dynamics of exchange rates. This is the reason why using survey data represents a real value-added providing a better description of the observed exchange rate. Public information, therefore, acts as a coordinating mechanism in accordance with the beauty contest analogy. We have intentionally integrated it into our framework to test its presence and intensity in the context of the exchange rate market. Furthermore, it also provides a complementary result to the empirical test of the scapegoat model posited by Bacchetta and Van Wincoop (2004) and implemented by Fratzscher et al. (2015), who find that using survey predictions on fundamentals as proxies for scapegoat effects improves our ability to explain exchange rate movements. Public information is, therefore, capable of capturing changes in actual exchange rate dynamics. This estimation discovers that the importance of higher order beliefs is associated with a larger weight to public information. On a rational level, predictors seek information on fundamentals. However, they end up attributing excess weight to public information which is not informative at least in the short run. This is clearly due, on one side, to the presence of higher order beliefs, and, on the other side, to the uncertainty entailed by the heterogeneity of expectations. The fundamental is therefore transformed into a scapegoat in the event of uncertainty regarding structural parameters. In particular, the higher value of the conditional variance of the fundamental, $\sigma_f$ (around 1.2) compared to the other variances, i.e., $\sigma_s, \sigma_x$, in Table 2 suggests that the short-term uncertainty proposed by Fratzscher et al. (2015) is coherent with uncertainty stemming from changes in fundamentals, specifically generating the scapegoat effect discussed by Bacchetta and Van Wincoop (2004).

We now move to the version of our model in which consistency restrictions are set. Results are reported in the first part of Table 2. We first observe that the coefficient $\delta$ is now 0.33. This suggests that although it is still relevant, the role of higher order beliefs is reduced when macroeconomic equilibrium is taken into account. Intuitively, in a model without consistency, *naive* agents choose their action in accordance with fundamental motive (their own personal choice) and higher order beliefs (the average of others) in a disconnected environment where they do not take into account the role played by market equilibrium. In a model with consistency, instead, investors are *sophisticated* and evaluate their choice within an integrated system including the evaluation of market equilibrium. Public information is still more informative about actions and beliefs of other players guiding the final realization of the exchange rates. However, its value is reduced around $53,7\%$. Albeit this estimate is smaller with respect to the former case, it is worth noting that the credibility interval is between 0.532 and 0.543. It does not include 0.50, and this suggests first that the data are extremely informative for this parameter, and second, that public information still plays a dominant role in agent's decisions. Moreover, we found that the information delivered by the market equilibrium increase even more the precisions of the signals in the consistent model, thus underlining the relevance of this source of information. Table 2 shows a large value of the precision of the private signal $\rho_x$, i.e., 7.35, compared to the values of $\rho_f$ and $\rho_s$, respectively of 4.8 and 4.74. This means that in the consistent model, private information appears to be more informative about individual expectation since a partial share of higher order beliefs is captured by the macroeconomic aggregation. We find that people still reason in higher orders exactly as in the previous case of non-consistency, but now bounded rationality reacts to an additional source of information at the aggregate level. Such source is exactly represented by the macroeconomic equilibrium. This is exactly the difference between the unconstrained model in which the perceived law of motion of eq. (14) is completely independent from the actual law of motion of eq. (12) compared to the constrained (consistent) model where the two laws of motion share the same moment distribution as explained in subsection 3.3.

Similar results summarized in Tables H.4 and H.5 in AppendixH are also available for *pound/usd* and *usd/yen* exchange rates. In particular, results show a relatively similar trend compared to the ones discussed for *eur/usd* rate and confirm the role of public and private information as well as the relevance of higher order beliefs in the analysis of exchange rates dynamics.

A further analysis relates to the ability of the model to predict actual exchange rates. We compare the one-step-ahead predictions provided by our model with the standard benchmark (Meese and Rogoff, 1983) for these applications, the random walk. Our one-step-ahead prediction have been computed as the average of the estimated predictive densities as shown in Figure 7.

Based on these predictions, we compute the Diebold and Mariano test. It shows that our one-step-ahead predictions are not worse than the random walk predictions and the $p$-value of this set of tests have been always larger than 90% considering both symmetric or asymmetric loss functions.

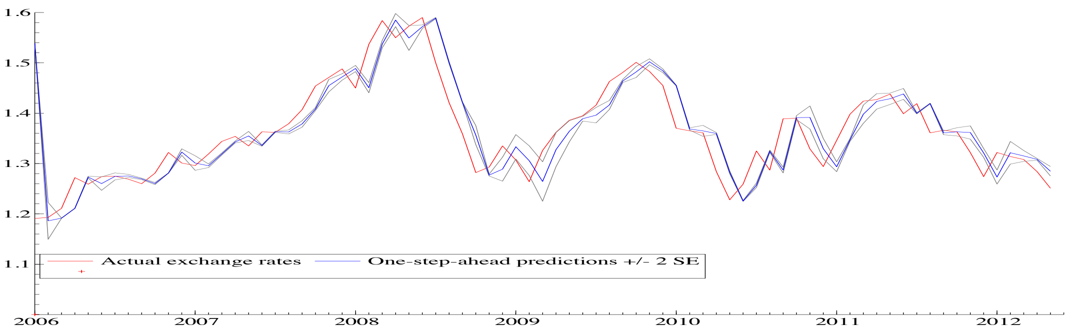


Figure 7

## 7. Conclusions

This paper shows the role of higher order beliefs in a learning game among investors.

In particular we look at an exchange rate market investigating how the impact of higher order beliefs may significantly vary according to the level of information that investors may take into account. Morris and Shin (2002) theoretically found that higher-order beliefs lead agents in a market to abandon highly informative but not commonly known private signals in favour of focal public ones. We extended this framework in a dynamic context observing whether higher order beliefs (and consequently public information) empirically have a crucial role when the optimization process relies on both individual evaluation and coordination among agents.

Our model was simple in concept. Formally, we have assumed that each individual has a quadratic-payoff function where agents wish to do the right thing matching the action to the fundamental, and do it together, i.e., coordinate with others' actions. The main justification for this was to focus the analysis on two novel aspects. First, the possibility to perfectly estimate the impact of the information disentangling the effect of both private and public sources. We thus were able to quantify the importance of each signal and to understand the weight that each agent devotes to others' opinion. Second, we observe the effect of evaluating macroeconomic news on agents' behaviour.

As discussed in detail in the Introduction, we verified consistency property connecting our learning game and the exchange rate dynamics. We imagined a market where agents perceive that their higher order beliefs on the real state of the economy are not enough to develop a correct forecasting rule. Such sophisticated agents search for additional sources trying to capture at least partially some information of the macroeconomic equilibrium. Results showed that when there is the potential for strong information at aggregate level, the values of higher order beliefs and public signal are reduced. Instead, the effect of higher order beliefs still remains preponderant whenever such information is no more available as in the case of naive investors.

Our test thus confirms the importance of higher order beliefs as suggested by several contributions in the literature. The take-home message, in particular, is that the importance of higher order beliefs is not absolute, but turns upon the type of information which investors face. Investors may welcome any information that helps them to resolve uncertainty about the state of the world and the likely actions of others. In this case, the role of higher order beliefs may be reduced due to the lower uncertainty of the restricted consistent environment. Each piece of information is partially included in the macroeconomic aggregation and this further decreases the public diffusion of news.

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## AppendixA. A stochastic determination of exchange rate

According to section 3, we now investigate a learning structure with an alternative characterization of exchange rate to accomodate for the features of the market microstructures. Due to the specific characteristics of the exchange rate market,we assume exchange rates are observed up to a measurement error, that is, they are a combination of the true exchange rate, i.e., a persistent and deterministic component, $s_t^*$, plus a measurement error, i.e., a transitory and stochastic one, $\epsilon_t^*$, describing the microstructure noise

$$s_t = s_t^* + \epsilon_t^* \qquad \epsilon_t^* \sim \mathcal{N}(0, \sigma_*^2) \tag{A.1}$$

The true exchange rates evolves according to

$$s_t^* = s_{t-1}^* + \gamma_t \quad \text{where} \quad \gamma_t \sim \mathcal{N}(0, \sigma_\gamma^2) \tag{A.2}$$

where the shock, $\gamma_t$, occurring at the beginning of period $t$ is normally distributed with mean 0, variance $\sigma_\gamma^2$, and precision $\rho_s \equiv \sigma_\gamma^{-2}$. The main reason of this extension is to demonstrate that the dynamics of the exchange rate and the equilibrium results substantially correspond to the case proposed in section 3, where the exchange rate is assumed to be ex-ante unobservable.

Denote $n = \{1, \ldots, i, ..., N\}$ as the finite series of predictors, where each agent $i$ observes noisy private and public signals about the exchange rate $s_t$. In particular, each agent $i$ receives a common public signal about the fundamental, $f_t$, as a function of the exchange rate:

$$f_t = s_t^* + \eta_t \qquad \eta_t \sim \mathcal{N}(0, \sigma_\eta^2) \tag{A.3}$$

and a private personal signal:

$$x_{it} = s_t^* + \epsilon_{it} \qquad \epsilon_{it} \sim \mathcal{N}(0, \sigma_\epsilon^2) \tag{A.4}$$

Agents compute the expected true exchange rate $s_t^*$. In this new setup, it makes sense to compute as a first step $\mathbb{E}[s_t^*|s_t, s_{t-1}, f_t, x_{it}]$, and then $\mathbb{E}[s_{t+1}^*|s_t, s_{t-1}, f_t, x_{it}]$. In this way we preserve the learning structure of the recursions. In short, we get the following state space model,

$$\begin{aligned} s_t &= s_t^* + \epsilon_t^* & \epsilon_t^* \sim \mathcal{N}(0, \sigma_*^2) \\ f_t &= s_t^* + \eta_t & \eta_t \sim \mathcal{N}(0, \sigma_\eta^2) \\ x_{it} &= s_t^* + \epsilon_{it} & \epsilon_{it} \sim \mathcal{N}(0, \sigma_\epsilon^2) \\ s_t^* &= s_{t-1}^* + \gamma_t^* & \gamma_t \sim \mathcal{N}(0, \sigma_\gamma^2) \end{aligned}$$

by running the standard Kalman recursions starting from $p(s_{t-1}^*|s_{t-1}) \sim \mathcal{N}(s_{t-1}, \sigma_*^2)$, we can show that

$$E[s_t^*|s_t, s_{t-1}, f_t, x_{it}] = \left(\frac{s_t}{\sigma_*^2} + \frac{f_t}{\sigma_\eta^2} + \frac{x_{it}}{\sigma_\epsilon^2} + \frac{s_{t-1}}{\sigma_*^2 + \sigma_\gamma^2}\right)\left(\frac{1}{\sigma_*^2} + \frac{1}{\sigma_\eta^2} + \frac{1}{\sigma_\epsilon^2} + \frac{1}{\sigma_*^2 + \sigma_\gamma^2}\right)^{-1} \quad \text{(A.5)}$$

whereas

$$Var[s_t^*|s_t, s_{t-1}, f_t, x_{it}] = \left(\frac{1}{\sigma_*^2} + \frac{1}{\sigma_\eta^2} + \frac{1}{\sigma_\epsilon^2} + \frac{1}{\sigma_*^2 + \sigma_\gamma^2}\right)^{-1} \quad \text{(A.6)}$$

The utility function is, therefore, modified as

$$U(e_{it}, \bar{e}_t, \sigma_e^2, s_{t+1}^*) = -(1-\delta)(e_{it} - s_{t+1}^*)^2 - \delta(e_{it} - \bar{e}_t)^2 \quad \text{(A.7)}$$

while the remaining part of the analysis purely follows the procedure in Subsection 3.1.

## AppendixB. Proof of Proposition 1

Solving the minimization problem in eq. (5) for $e_{it}$, we obtain that:

$$e_{it}(x_{it}; f_t; \rho_{\tilde{y}}; \rho_x) = (1-\delta)\mathbb{E}_t^i[s_{t+1}|x_{it}; f_t; \rho_{\tilde{y}}; \rho_x] + \delta\mathbb{E}_t^i[\bar{e}_t|x_{it}; f_t; \rho_{\tilde{y}}; \rho_x] \quad \text{(B.1)}$$

which can be rewritten as:

$$e_{it}(x_i; f; \rho_{\tilde{y}}; \rho_x) = (1-\delta)\mathbb{E}^i_t[s_{t+1}|x_{it}; f_t; \rho_{\tilde{y}}; \rho_x] + \delta\frac{e_{it}}{n} + \delta\frac{n-1}{n}\mathbb{E}^i_t[e_{-it}|x_{it}; f_t; \rho_{\tilde{y}}; \rho_x] \quad \text{(B.2)}$$

where $\mathbb{E}^i_t[e_{-it}|x_{it}; f_t; \rho_{\tilde{y}}; \rho_x] = \mathbb{E}^i_t[(\frac{e_{1t}+...+e_{i-1t}+e_{i+1t}+...e_{nt}}{n})|x_{it}; f_t; \rho_{\tilde{y}}; \rho_x]$. In the unique equilibrium with heterogeneous information, each individual $i \neq j$ at time $t$ follows a linear strategy as:

$$e_{it}(x_{it}; f_t; \rho_{\tilde{y}}; \rho_x) = \varphi_x x_{it} + \varphi_{\tilde{y}}\tilde{y}_t \quad \text{(B.3)}$$

According to this strategy, the predictor's expectation about the other $(n-1)$ agents is linear in $(f_t; s_{t+1})$ and is given by:

$$\mathbb{E}^i_t[e_{-it}|x_{it}; f_t; \rho_{\tilde{y}}; \rho_x] = \varphi_x\mathbb{E}^i_t[x_{-it}] + \varphi_{\tilde{y}}\tilde{y}_t$$

then according to eq. (3), $\mathbb{E}^i_t[x_{-it}] = \mathbb{E}^i_t[s_t + \epsilon_{-it}]$. Moreover since $\epsilon_{it} \sim \mathcal{N}(0, \sigma^2_\epsilon)$ and using it into eq. (1), $\mathbb{E}^i_t[s_t + \epsilon_{-it}] = \mathbb{E}^i_t[s_{t+1}]$. Therefore,

$$\mathbb{E}^i_t[e_{-it}|x_{it}; f_t; \rho_{\tilde{y}}; \rho_x] = \varphi_x\mathbb{E}^i_t[s_{t+1}] + \varphi_{\tilde{y}}\tilde{y}_t \quad \text{(B.4)}$$

Plugging eq. (B.3) and eq. (B.4) into eq. (B.2),

$$e_{it} = (1-\delta)\mathbb{E}^i_t[s_{t+1}|x_{it}; f_t; \rho_{\tilde{y}}; \rho_x] + \delta\frac{e_{it}}{n} + \delta\frac{n-1}{n}\mathbb{E}^i_t[\varphi_x s_{t+1} + \varphi_{\tilde{y}}\tilde{y}_t|x_{it}; f_t; \rho_{\tilde{y}}; \rho_x] \quad \text{(B.5)}$$

while rearranging it,

$$e_{it} = (1-\varrho+\varrho\varphi_x)\left[\frac{\rho_x}{\rho_{\tilde{y}}+\rho_x}x_{it} + \frac{\rho_{\tilde{y}}}{\rho_{\tilde{y}}+\rho_x}\tilde{y}_t\right] + \varrho\varphi_{\tilde{y}}\tilde{y}_t \quad \text{(B.6)}$$

where $\varrho = \frac{n\delta-\delta}{n-\delta}$. According to eq. (B.3), the coefficients $(\varphi_x; \varphi_{\tilde{y}})$ for the optimal linear strategy must therefore satisfy,

$$\varphi_x = \frac{(1-\varrho)\rho_x}{(1-\varrho)\rho_x+\rho_{\tilde{y}}} \quad \text{and} \quad \varphi_{\tilde{y}} = \frac{\rho_{\tilde{y}}}{(1-\varrho)\rho_x+\rho_{\tilde{y}}} \quad \text{(B.7)}$$

as the unique solution of the system.

## AppendixC. Proof of eq. (9)

We briefly mention the main steps to determine the average interest rate parity condition proposed in eq. (9). This framework represents one of the workhorse frameworks in noisy rational expectations literature and can be derived by a standard utility model with a single risky asset à la Hellwig (1980). The demand for foreign bonds by investor $i$, namely, $b^i_{F_t}$, is:

$$b^i_{F_t} = \frac{\mathbb{E}^i_t[s_{t+1}] - s_t + i^*_t - i_t}{\gamma \sigma^2_t} - b_{it} \tag{C.1}$$

where the individual expectation of the next period exchange rate is $\mathbb{E}^i_t[s_{t+1}]$, while $\sigma^2_t$ is the conditional variance and $b_{it}$ is the hedge against non-asset income. Since bonds are in a zero net supply, market equilibrium is given by,

$$\int_0^1 b^i_{F_t} \, di = 0 \tag{C.2}$$

The exchange rate exposure is assumed to be equal to the average exposure plus an idiosyncratic term, i.e., $b_{it} = b_t + \epsilon_{it}$ where $\epsilon_{it} \sim \mathcal{N}(0, \sigma^2_\epsilon)$. Market equilibrium determines the following interest rate arbitrage condition:

$$\bar{\mathbb{E}}_t[s_{t+1}] - s_t = i_t - i^*_t + \psi_t \tag{C.3}$$

where $\bar{\mathbb{E}}_t[s_{t+1}]$ is the average expectation across all investors, while $\psi_t = \gamma \sigma^2_t b_t$ is defined as the expectational error or a risk premium associated with liquidity or hedge trade.

## AppendixD. Alternative money market model

In the spirit of Bacchetta and Van Wincoop (2006), we now propose a simple alternative model of exchange rate. Our results are robust to variations to the macroeconomic framework and substantially coincide to the ones proposed in Subsection 3.2. Here, the main difference is that we do not use a Taylor rule on interest rate differential.

Let us start by the standard dynamic two-country model of exchange rates with the following basic relationships. Define, $p_t$ and $p_t^*$, as the logs of the Home and the Foreign price level, respectively, and $s_t$ is the log-exchange rate. The purchasing power parity equation is, then, ensured as:

$$p_t = p_t^* + s_t \tag{D.1}$$

A standard money market equilibrium is realized such that,

$$m_t - p_t = \phi y_t - \alpha i_t, \tag{D.2}$$

$$m_t^* - p_t^* = \phi y_t^* - \alpha i_t^* \tag{D.3}$$

where $m_t$ and $m_t^*$ are the logs-money supplies, $y_t$ and $y_t^*$ are the logs-output levels, while $i_t$ and $i_t^*$ are the interest rates, respectively for Home and Foreign countries.[22] The observable fundamental $f_t$ is now defined as a function of money supplies and output gap:

$$f_t = m_t - m_t^* - \phi(y_t - y_t^*) \tag{D.4}$$

By combining (C.3-D.4), the aggregate expectation is implemented in a standard equilibrium model for exchange rates with heterogeneous agents:

$$s_t = \lambda \bar{\mathbb{E}}_t[s_{t+1}] + (1-\lambda) f_t - \lambda \psi_t \tag{D.5}$$

where $\lambda = \alpha/(1+\alpha)$, $\alpha$ is the parameter describing interest rates in equilibrium and $\psi_t$ is the liquidity premium.

The combination of learning process in a macroeconomic setup allows to easily derive a

[22]We assume that there is no a-priori differences in the structure of the two countries, then $\phi$ and $\alpha$ are equal among them.

statistical model necessary to make inference on utility function's parameters. Eq. (D.5) explains how the current exchange rate is related in a simple way to the heterogeneous expectations of investors, $\bar{\mathbb{E}}_t[s_{t+1}]$, a commonly observed fundamental, $f_t$, and the value of liquidity trade, $\psi_t$.

According to the individual solution of the learning game, $\mathbb{E}_t^i(s_{t+1}) = \varphi_x x_{it} + \varphi_y \tilde{y}_t$, again the aggregation of the individual predictions of $n$ investors is,

$$\bar{\mathbb{E}}_t[s_{t+1}] = \varphi_x \bar{x}_t + \varphi_y \tilde{y}_t, \tag{D.6}$$

while, by substituting it in eq.(D.5)

$$s_t = \lambda(\varphi_x \bar{x}_t + \varphi_{\tilde{y}} \tilde{y}) + (1-\lambda) f_t - \lambda \psi_t. \tag{D.7}$$

Rearranging D.7, we get:

$$s_t = (1 - \lambda + \lambda \tau_2 \varphi_{\tilde{y}}) f_t + \lambda \varphi_x \bar{x}_t + \lambda \tau_1 \varphi_{\tilde{y}} s_{t-1} - \lambda \psi_t, \tag{D.8}$$

where $\tau_1 = \frac{\rho_s}{\rho_s + \rho_f}$ and $\tau_2 = \frac{\rho_f}{\rho_s + \rho_f}$. Therefore, eq. (D.8) can be reduced to:

$$s_t = \beta_1 f_t + \beta_2 \bar{x}_t + \beta_3 s_{t-1} + \beta_4 \psi_t, \tag{D.9}$$

nwhere $\beta_1 = (1 - \lambda + \lambda \tau_2 \varphi_{\tilde{y}})$, $\beta_2 = \lambda \varphi_x$, $\beta_3 = \lambda \tau_1 \varphi_{\tilde{y}}$ and $\beta_4 = -\lambda$. The structure and the meaning of the variables are similar to the ones proposed in the main text.

## AppendixE. Proof of eqs. (15), (16), (17) and (18)

Let us start by minimizing the least square estimator as:

$$\sum_{i=1}^{n} \left[ s_t - a s_{t-1} - b f_t - (1-a-b) \bar{x}_t \right]^2 \tag{E.1}$$

then deriving from $a$ and $b$,

$$\begin{cases} \frac{\partial}{\partial a} & = & 2\sum_{i=1}^{n}\left[s_t - as_{t-1} - bf_t - (1-a-b)\bar{x}_t)\right]\left(\bar{x}_t - s_{t-1}\right) \\ \frac{\partial}{\partial b} & = & 2\sum_{i=1}^{n}\left[s_t - as_{t-1} - bf_t - (1-a-b)\bar{x}_t)\right]\left(\bar{x}_t - f_t\right) \end{cases} \tag{E.2}$$

After some tedious steps,

$$\hat{a} = \frac{Var\left(f_t - \bar{x}_t\right)COV\left(s_t - \bar{x}_t, s_{t-1} - \bar{x}_t\right) - COV\left(s_t - \bar{x}_t, f_t - \bar{x}_t\right)COV\left(s_{t-1} - \bar{x}_t, f_t - \bar{x}_t\right)}{Var\left(f_t - \bar{x}_t\right)Var\left(s_t - \bar{x}_t\right) - COV^2\left(s_t - \bar{x}_t, f_t - \bar{x}_t\right)} \tag{E.3}$$

$$\hat{b} = \frac{Var\left(s_{t-1} - \bar{x}_t\right)COV\left(s_t - \bar{x}_t, f_t - \bar{x}_t\right) - COV\left(s_t - \bar{x}_t, s_{t-1} - \bar{x}_t\right)COV\left(s_{t-1} - \bar{x}_t, f_t - \bar{x}_t\right)}{Var\left(f_t - \bar{x}_t\right)Var\left(s_{t-1} - \bar{x}_t\right) - COV^2\left(s_{t-1} - \bar{x}_t, f_t - \bar{x}_t\right)} \tag{E.4}$$

Solving (E.3) and (E.4) through eqs. (1-2) and eq. (12), while aggregating the private signals, $\bar{x}_t = s_t + \gamma_t$, we obtain the following forecasting parameters of eq. (14),

$$\hat{a} = \frac{\lambda\rho_s + (1 - \rho_s(1 - \varphi_{\tilde{y}}) + \varphi_{\tilde{y}} + n\rho_f\varphi_{\tilde{y}}}{\rho_s(2\lambda + \rho_f + \rho_s) + n(\lambda\rho_f + \rho_s)\rho_x} \tag{E.5}$$

$$\hat{b} = \frac{\lambda(2\rho_s\rho_f + n\rho_s\rho_x) + \lambda((\rho_f^2 - \rho_f\rho_s)\tau_1 - \rho_s\rho_f - n\rho_s\rho_x - \rho_f^2)\varphi_{\tilde{y}}}{\rho_f(\rho_f + \rho_s + n\rho_x)} \tag{E.6}$$

## AppendixF. Proof of Proposition 2

Proposition 2 solves the correspondence between the forecasting parameters and the market equilibrium as in eqs. (19) and (20), respectively,

$$\hat{a} = \varphi_y\tau_1 \text{ and } \hat{b} = \varphi_y\tau_2 \tag{F.1}$$

as mentioned in Definition 1. Starting by eqs. (E.5) and (E.6), we may rewrite the conditions of (F.1) as follows:

$$\frac{\lambda\rho_s + (1-\rho_s(1-\varphi_{\tilde{y}}) + \varphi_{\tilde{y}} + n\rho_f\varphi_{\tilde{y}}}{\rho_s(2\lambda+\rho_f+\rho_s) + n(\lambda\rho_f+\rho_s)\rho_x} = \varphi_y\tau_1 \tag{F.2}$$

$$\frac{\lambda(2\rho_s\rho_f + n\rho_s\rho_x) + \lambda((\rho_f^2 - \rho_f\rho_s)\tau_1 - \rho_s\rho_f - n\rho_s\rho_x - \rho_f^2)\varphi_{\tilde{y}}}{\rho_f(\rho_f+\rho_s+n\rho_x)} = \varphi_y\tau_2 \tag{F.3}$$

However, since $\tau_1 = \frac{\rho_s}{\rho_s+\rho_f}$ and $\varphi_y = \frac{\rho_s+\rho_f}{(1-\varrho)\rho_x+(\rho_s\rho_f)}$, we substitute these values of $\tau_1$ and $\varphi_y$ into eqs. (F.2) and (F.3) obtaining a system of recursive equations with the following solutions of the signal precisions $\rho_s$ and $\rho_f$,

$$\rho_s = \frac{(n-\delta)\rho_f^2(\rho_f+\lambda\rho_f+n\rho_x)}{(1-\delta)(2n\lambda\rho_f\rho_x + n^2\lambda\rho_x^2 - (n-\delta)\rho_f^2)} \tag{F.4}$$

$$\rho_f = \frac{\delta(2\lambda-\rho_s) + n((\rho_s-2) - \lambda\delta + \lambda(\rho_s-\rho_x)(\delta-\lambda)) + \Phi}{2n(n-\delta)\lambda} \tag{F.5}$$

where

$$\begin{aligned}\Phi &= ((n-\delta)(4n\lambda((n-\delta)\rho_s^2 + n((n+\lambda-\delta(1+\lambda)\rho_s)\rho_x \\ &\quad -(1-\delta)\lambda + (n-\delta)(\rho_s - \lambda(2-n(\rho_x-\rho_s)))^2))^{\frac{1}{2}}\end{aligned}$$

These eqs. correspond to eqs. (21)-(23) in Subsection 3.3.

## AppendixG. Consistent solutions for $n \to \infty$

We follow the procedure pointed out in subsection 3.3. The first step is to find the simplified version of the forecasting parameters of eqs. (17) and (18), when $n \to \infty$. It follows that:

$$\hat{a} = \frac{\lambda\rho_f}{\rho_s+\lambda\rho_f}\varphi_y \quad \text{and} \quad \hat{b} = \frac{\lambda\rho_s(1-\varphi_y)}{\rho_f} \tag{G.1}$$

Then, since from eqs. (19) and (20), $\hat{a} = \varphi_y\tau_1$ and $\hat{b} = \varphi_y\tau_2$, while $\tau_1 = \frac{\rho_s}{\rho_s+\rho_f}$ and $\varphi_y = \frac{\rho_s+\rho_f}{(1-\varrho)\rho_x+(\rho_s\rho_f)}$, the analytical solution of the recursive system in (G.1) results in,

$$\begin{cases} \rho_s &= (1-\delta)\lambda^2 \rho_x \\ \rho_f &= (1-\delta)\lambda^{\frac{3}{2}} \rho_x \end{cases} \tag{G.2}$$

where the precision of the exchange rate $\rho_s$ and the precision of the public signal $\rho_f$ solely depend on the weighted precision of the private signal $\rho_x$. We show that the weight of private information is positive,

$$\varphi_x = \frac{1}{1 + \lambda^{\frac{3}{2}} + \lambda^2}$$

confirming the impact of the private information is still in place, independently of $n$. Intuitively, although there is a large number of agents such that the effect of private information at the aggregate level coincides with the exchange rate, each agent still has her own private information that clearly influences the dynamics of the other variables and consequently impacts other agents' beliefs and the dynamics of exchange rate.

## AppendixH. Other datasets

Here we report some empirical results for other exchange rates, and in particular we focus on the pound/usd and the usd/yen case. For each exchange rate, we considered a panel of 15 agents observed from January 2006 to June 2012. More details on these dataset are reported in Section 4.

Table H.4: Pound/usd: Posterior computation (MCMC) - Structural parameters. Note that the prior distributions are defined through their averages and variances in parenthesis.

| | Consistency | | | Non Consistency | | |
|---|---|---|---|---|---|---|
| | Mean | 95% Cred. Int. | Prior | Mean | 95% Cred. Int. | Prior |
| $\beta_1$ | -0.5485 | [-0.5537,-0.5433] | | -0.5808 | [-0.6438,-0.5114] | |
| $\beta_2$ | 0.4382 | [0.4329,0.4435] | | 0.0773 | [0.0635,0.0922] | |
| $\beta_3$ | 0.4436 | [0.4392,0.4481] | | 0.6129 | [0.5541,0.6776] | |
| $\beta_4$ | -0.9621 | [-0.9738,-0.948] | | -0.8906 | [-0.9607,-0.7801] | |
| $\rho_f$ | 3.7579 | [3.4096,4.1526] | | 1.7177 | [1.4349,2.0267] | $\mathcal{G}(1.0, 0.1)$ |
| $\rho_s$ | 3.6861 | [3.3372,4.0808] | | 2.793 | [2.4535,3.1514] | $\mathcal{G}(1.0, 0.1)$ |
| $\rho_x$ | 6.1433 | [5.6616,6.6797] | $\mathcal{G}(4.0, 0.2)$ | 2.6294 | [2.3352,2.9518] | $\mathcal{G}(4.0, 0.2)$ |
| $\lambda$ | 0.9621 | [0.948,0.9738] | $\mathcal{B}(0.67, 0.05)$ | 0.8906 | [0.7801,0.9607] | $\mathcal{B}(095, 0.025)$ |
| $\delta$ | 0.3677 | [0.3288,0.4077] | $\mathcal{B}(0.5, 0.025)$ | 0.8457 | [0.814,0.8739] | $\mathcal{B}(0.5, 0.025)$ |
| $\varphi_{\tilde{y}}$ | 0.5445 | [0.5381,0.5507] | | 0.9132 | [0.8996,0.9261] | |
| $\varphi_x$ | 0.4555 | [0.4493,0.4619] | | 0.0868 | [0.0739,0.1004] | |
| $\sigma_s$ | 0.4848 | [0.2528,1.1437] | $\mathcal{IG}(0.6, 0.2)$ | 0.2197 | [0.1772,0.2665] | $\mathcal{IG}(0.6, 0.2)$ |
| $\sigma_\phi$ | 0.2184 | [0.1779,0.2999] | $\mathcal{IG}(0.6, 0.2)$ | 0.2372 | [0.1904,0.2842] | $\mathcal{IG}(0.6, 0.2)$ |
| $\theta_f$ | -0.1795 | [-0.1916,-0.1671] | | -0.4654 | [-0.5043,-0.426] | |
| $\sigma_f$ | 2.3615 | [2.2843,2.4465] | | 1.1217 | [1.061,1.1854] | |
| $\theta_x$ | -0.1248 | [-0.1328,-0.1168] | | -0.3923 | [-0.4191,-0.3653] | |
| $\sigma_x$ | 1.1427 | [1.1322,1.1532] | | 0.9861 | [0.9415,1.03] | |

Table H.5: Usd/yen: Posterior computation (MCMC) - Structural parameters. Note that the prior distributions are defined through their averages and variances in parenthesis.

| | Consistency | | | Non Consistency | | |
|---|---|---|---|---|---|---|
| | Mean | 95% Cred. Int. | Prior | Mean | 95% Cred. Int. | Prior |
| $\beta_1$ | -0.5505 | [-0.5565,-0.5447] | | -0.7219 | [-0.7944,-0.624] | |
| $\beta_2$ | 0.4349 | [0.4283,0.4406] | | 0.0444 | [0.037,0.0517] | |
| $\beta_3$ | 0.4411 | [0.4357,0.4457] | | 0.7931 | [0.7559,0.8313] | |
| $\beta_4$ | -0.9588 | [-0.9725,-0.9422] | | -0.8844 | [-0.9615,-0.7632] | |
| $\rho_f$ | 3.5417 | [3.1602,3.9245] | | 0.9252 | [0.7549,1.1051] | $\mathcal{G}(1, 0.1)$ |
| $\rho_s$ | 3.4682 | [3.085,3.8505] | | 3.8571 | [3.5204,4.2179] | $\mathcal{G}(1, 0.1)$ |
| $\rho_x$ | 5.8672 | [5.407,6.3281] | $\mathcal{G}(4.0, 0.2)$ | 2.6278 | [2.3393,2.9323] | $\mathcal{G}(4, 0.2)$ |
| $\lambda$ | 0.9588 | [0.9422,0.9725] | $\mathcal{B}(0.67, 0.05)$ | 0.8844 | [0.7632,0.9615] | $\mathcal{B}(0.95, 0.025)$ |
| $\delta$ | 0.3731 | [0.3323,0.4151] | $\mathcal{B}(0.5, 0.025)$ | 0.9094 | [0.8932,0.9241] | $\mathcal{B}(0.5, 0.025)$ |
| $\varphi_{\tilde{y}}$ | 0.5464 | [0.5392,0.5535] | | 0.9498 | [0.9432,0.9555] | |
| $\varphi_x$ | 0.4536 | [0.4465,0.4608] | | 0.0502 | [0.0445,0.0568] | |
| $\sigma_s$ | 0.2304 | [0.1902,0.2822] | $\mathcal{IG}(0.6, 0.2)$ | 0.2147 | [0.1791,0.2614] | $\mathcal{IG}(0.6, 0.2)$ |
| $\sigma_\phi$ | 0.2353 | [0.192,0.2895] | $\mathcal{IG}(0.6, 0.2)$ | 0.2364 | [0.191,0.2935] | $\mathcal{IG}(0.6, 0.2)$ |
| $\theta_f$ | -0.187 | [-0.2017,-0.1739] | | -0.616 | [-0.6464,-0.5863] | |
| $\sigma_f$ | 2.3136 | [2.2268,2.3981] | | 1.3289 | [1.2365,1.4363] | |
| $\theta_x$ | -0.1294 | [-0.1376,-0.1219] | | -0.4478 | [-0.4726,-0.424] | |
| $\sigma_x$ | 1.1486 | [1.1388,1.1595] | | 0.923 | [0.8871,0.9605] | |